Supplementary Document for "Uncertainty Quantification of Stochastic Simulation for Black-box Computer Experiments"

Youngjun Choe · Henry Lam · Eunshin Byon

This supplementary document contains the proofs of Propositions 1–3, Lemmas 1–2, and Corollary 1, as well as the implementation details on the NREL simulators.

S1 Proof of Proposition 1

Suppose that $s_y(\mathbf{x}) \neq 0$ implies $\hat{s}_y(\mathbf{x}) \neq 0$ in the support of f. Under this condition, we prove that an equivalent statement of Assumption 1, namely, $s_y(\mathbf{x})f(\mathbf{x}) \neq 0 \Rightarrow q(\mathbf{x}) \neq 0$ for any $\mathbf{x}$, holds.

The statement, $s_y(\mathbf{x})f(\mathbf{x}) \neq 0$, implies that $s_y(\mathbf{x}) \neq 0$ and $f(\mathbf{x}) \neq 0$. Then, it follows that $\hat{s}_y(\mathbf{x}) \neq 0$ and $f(\mathbf{x}) \neq 0$, and then

$$q(\mathbf{x}) = \frac{1}{\hat{C}_q} f(\mathbf{x}) \sqrt{\frac{1}{n} \hat{s}_y(\mathbf{x}) \left(1 - \hat{s}_y(\mathbf{x})\right) + \hat{s}_y(\mathbf{x})^2}$$

in (5) is not zero for $n \geq 1$. Therefore, if $s_y(\mathbf{x}) \neq 0$ implies $\hat{s}_y(\mathbf{x}) \neq 0$ in the support of f, then Assumption 1 holds. $\square$

S2 Proof of Proposition 2

We will use the property, $\mathbb{E}_q(L) = \mathbb{E}_f(1)$, that holds under Assumption 1 in the subsequent derivation.

Y. Choe
Department of Industrial and Systems Engineering, University of Washington,
3900 E Stevens Way NE, Seattle, WA 98195, USA
E-mail: ychoe@uw.edu

H. Lam
Department of Industrial Engineering and Operations Research, Columbia University,
500 W. 120th Street, New York, NY 10027, USA

E. Byon
Department of Industrial and Operations Engineering, University of Michigan,
1205 Beal Avenue, Ann Arbor, MI 48109, USA

By plugging the SIS density, $q_y(\mathbf{x})$, in (5) into $\mathbb{E}_q\left[\mathbb{I}(Y>y)L^2\right]$ leads to

$$
\begin{aligned}
\mathbb{E}_q\left[\mathbb{I}(Y>y)L^2\right] &= \mathbb{E}_q\left[\mathbb{E}\left[\mathbb{I}(Y>y)\mid \mathbf{X}\right]L^2\right] \\
&= \mathbb{E}_q\left[s_y(\mathbf{X})\,L^2\right] \\
&= \mathbb{E}_f\left[s_y(\mathbf{X})\,\frac{f(\mathbf{X})}{q_y(\mathbf{X})}\right] \\
&= \mathbb{E}_f\left[s_y(\mathbf{X})\,\frac{f(\mathbf{X})}{\frac{1}{\hat{C}_q}f(\mathbf{x})\sqrt{\frac{1}{n}\hat{s}_y(\mathbf{x})\left(1-\hat{s}_y(\mathbf{x})\right)+\hat{s}_y(\mathbf{x})^2}}\right] \\
&= \hat{C}_q\mathbb{E}_f\left[\frac{s_y(\mathbf{X})}{\sqrt{\frac{1}{n}\hat{s}_y(\mathbf{X})\left(1-\hat{s}_y(\mathbf{X})\right)+\hat{s}_y(\mathbf{X})^2}}\right] \\
&\leq \hat{C}_q\mathbb{E}_f\left[\frac{s_y(\mathbf{X})}{\sqrt{\hat{s}_y(\mathbf{X})^2}}\right] && \text{(S2.1)} \\
&< \infty. && \text{(S2.2)}
\end{aligned}
$$

The inequality in (S2.1) holds because $\frac{1}{n}s_y(\mathbf{X})\left(1-s_y(\mathbf{X})\right)\geq 0$. The inequality in (S2.2) holds under the stated condition, $\mathbb{E}_f[s_y(\mathbf{X})/\hat{s}_y(\mathbf{X})]<\infty$, because

$$
\begin{aligned}
\hat{C}_q &= \int_{\mathcal{X}_f} f(\mathbf{x})\sqrt{\frac{1}{n}\hat{s}_y(\mathbf{x})\cdot\left(1-\hat{s}_y(\mathbf{x})\right)+\hat{s}_y(\mathbf{x})^2}\,\mathrm{d}\mathbf{x} \\
&\leq \int_{\mathcal{X}_f} f(\mathbf{x})\sqrt{(1+1)}\,\mathrm{d}\mathbf{x} && \text{(S2.3)} \\
&= \sqrt{2} \\
&< \infty,
\end{aligned}
$$

where $\mathcal{X}_f$ is the support of f. The inequality in (S2.3) holds because the both summands within the square root are bounded above by 1 for $n \geq 1$. Therefore, the SIS density, $q_y(\mathbf{x})$, in (5) satisfies Assumption 2 if $\mathbb{E}_f[s_y(\mathbf{X})/\hat{s}_y(\mathbf{X})]<\infty$. □

S3 Proof of Proposition 3

To show $\mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right] < \infty$ for $q = q_y$, we bound $\mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]$ from above by a constant:

$$
\begin{aligned}
\mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right] &= \mathbb{E}_f\left[\tilde{h}(\mathbf{X})\frac{1}{\hat{C}_q}\sqrt{\frac{1}{n}\hat{s}_y(\mathbf{X})\left(1-\hat{s}_y(\mathbf{X})\right)+\hat{s}_y(\mathbf{X})^2}\right] \\
&\le \frac{1}{\hat{C}_q}\mathbb{E}_f\left[\tilde{h}(\mathbf{X})\sqrt{\hat{s}_y(\mathbf{X})}\right] && \text{(S3.4)} \\
&= \frac{1}{\hat{C}_q}\mathbb{E}_f\left[\sqrt{\frac{1-\hat{s}_y(\mathbf{X})}{\hat{s}_y(\mathbf{X})}}\sqrt{\hat{s}_y(\mathbf{X})}\right] \\
&= \frac{1}{\hat{C}_q}\mathbb{E}_f\left[\sqrt{1-\hat{s}_y(\mathbf{X})}\right] \\
&\le \frac{1}{\hat{C}_q} && \text{(S3.5)} \\
&< \infty.
\end{aligned}
$$

The inequality in (S3.4) holds for $n \ge 1$. The inequality in (S3.5) holds because $\hat{s}_y(\mathbf{x}) \ge 0$. Because q_y is a density function, its normalizing constant, $\hat{C}_q$, is a positive constant. □

S4 Proof of Lemma 1

To prove $N_k \xrightarrow{P} \tilde{N}_k$ in (13), we first define

$$\eta_k \equiv n\frac{h_n(\mathbf{X}_k)}{\sum_{j=1}^m h_n(\mathbf{X}_j)} + \frac{1}{2}, \tag{S4.1}$$

$$\tilde{\eta}_k \equiv \frac{\tilde{h}(\mathbf{X}_k)}{c_0\mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]} + \frac{1}{2},$$

and

$$r(x) \equiv \max\left(1, \lfloor x \rfloor\right), \tag{S4.2}$$

so that N_k in (6) and $\tilde{N}_k$ in (14) can be expressed as

$$
\begin{aligned}
N_k &= \max\left(1, \left\lfloor n\frac{h_n(\mathbf{X}_k)}{\sum_{j=1}^m h_n(\mathbf{X}_j)} + \frac{1}{2}\right\rfloor\right) \\
&= r(\eta_k)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{N}_k &= \max\left(1, \left\lfloor \frac{\tilde{h}(\mathbf{X}_k)}{c_0\mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]} + \frac{1}{2}\right\rfloor\right) \\
&= r(\tilde{\eta}_k)\,,
\end{aligned}
$$

respectively.

Next, we prove $\eta_k \xrightarrow{P} \tilde{\eta}_k$ and then $r(\eta_k) \xrightarrow{P} r(\tilde{\eta}_k)$ to complete the proof.

- Proof of $\eta_k \xrightarrow{P} \tilde{\eta}_k$: Note that η_k in (S4.1) can be expressed as

$$\begin{aligned}\eta_k &= n\frac{h_n(\mathbf{X}_k)}{\sum_{j=1}^m h_n(\mathbf{X}_j)} + \frac{1}{2} \\ &= \frac{h_n(\mathbf{X}_k)}{(c_0 + o(1))\frac{1}{m}\sum_{j=1}^m h_n(\mathbf{X}_j)} + \frac{1}{2} \qquad \text{(S4.3)}\end{aligned}$$

by Assumption 3. To later invoke the continuous mapping theorem, we first prove that both the denominator and the numerator of the first term in (S4.3) converge in probability.

For the denominator of the first term in (S4.3), we want to prove

$$\frac{1}{m}\sum_{j=1}^m h_n(\mathbf{X}_j) \xrightarrow{P} \mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right] \qquad \text{(S4.4)}$$

as $m \to \infty$. Thus, for any $\epsilon > 0$, consider the probability

$$\begin{aligned}&\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^m h_n(\mathbf{X}_j) - \mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]\right| > \epsilon\right) \\ &= \mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^m h_n(\mathbf{X}_j) - \frac{1}{m}\sum_{j=1}^m \tilde{h}(\mathbf{X}_j) + \frac{1}{m}\sum_{j=1}^m \tilde{h}(\mathbf{X}_j) - \mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]\right| > \epsilon\right) \\ &\le \mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^m h_n(\mathbf{X}_j) - \frac{1}{m}\sum_{j=1}^m \tilde{h}(\mathbf{X}_j)\right| + \left|\frac{1}{m}\sum_{j=1}^m \tilde{h}(\mathbf{X}_j) - \mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]\right| > \epsilon\right) \\ &\le \mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^m h_n(\mathbf{X}_j) - \frac{1}{m}\sum_{j=1}^m \tilde{h}(\mathbf{X}_j)\right| > \epsilon/2\right) + \mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^m \tilde{h}(\mathbf{X}_j) - \mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]\right| > \epsilon/2\right), \\ &\qquad \text{(S4.5)}\end{aligned}$$

where the second term in (S4.5) converges to zero as $m \to \infty$ by the weak law of large numbers because $\tilde{h}(\mathbf{X}_j), j = 1, \ldots, m$, are i.i.d. random variables with a finite mean of $\mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]$ by the condition in (9) in Assumption 3.

The first term in (S4.5) can be expressed as

$$\begin{aligned}&\mathbb{P}\left(\left|\frac{1}{m}\sum_{j=1}^m h_n(\mathbf{X}_j) - \frac{1}{m}\sum_{j=1}^m \tilde{h}(\mathbf{X}_j)\right| > \epsilon/2\right) \qquad \text{(S4.6)} \\ &= \mathbb{P}\left(\frac{1}{m}\sum_{j=1}^m \left(\tilde{h}(\mathbf{X}_j) - h_n(\mathbf{X}_j)\right) > \epsilon/2\right) \qquad \text{(S4.7)} \\ &\le \frac{1}{\epsilon/2}\mathbb{E}_q\left[\frac{1}{m}\sum_{j=1}^m \left(\tilde{h}(\mathbf{X}_j) - h_n(\mathbf{X}_j)\right)\right] \qquad \text{(S4.8)} \\ &= \frac{1}{\epsilon/2}\left(\mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right] - \mathbb{E}_q[h_n(\mathbf{X})]\right), \qquad \text{(S4.9)}\end{aligned}$$

where the equality in (S4.7) holds because

$$\begin{aligned}
h_n(\mathbf{X}_j) &= \sqrt{\frac{n\left(1-\hat{s}_y(\mathbf{X}_j)\right)}{1+(n-1)\,\hat{s}_y(\mathbf{X}_j)}} \\
&= \sqrt{\frac{1-\hat{s}_y(\mathbf{X}_j)}{\left(1-\hat{s}_y(\mathbf{X}_j)\right)/n+\hat{s}_y(\mathbf{X}_j)}} \\
&\le \sqrt{\frac{1-\hat{s}_y(\mathbf{X}_j)}{\hat{s}_y(\mathbf{X}_j)}} \\
&= \tilde{h}(\mathbf{X}_j)
\end{aligned}$$

for $j = 1, \dots, m$. The inequality in (S4.8) holds by Markov's inequality. The equality in (S4.9) holds because $\mathbf{X}_j, j = 1, \dots, m$ are identically distributed. The dominated convergence theorem yields that $\mathbb{E}_q[h_n(\mathbf{X})] \to \mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]$ as $n \to \infty$, because $|h_n(\mathbf{x})| \le \tilde{h}(\mathbf{x})$ for $n \ge 1$, and

$$\begin{aligned}
\lim_{n\to\infty} h_n(\mathbf{x}) &= \lim_{n\to\infty} \sqrt{\frac{n\left(1-\hat{s}_y(\mathbf{x})\right)}{1+(n-1)\,\hat{s}_y(\mathbf{x})}} \\
&= \lim_{n\to\infty} \sqrt{\frac{1-\hat{s}_y(\mathbf{x})}{1/n+(1-1/n)\,\hat{s}_y(\mathbf{x})}} \\
&= \sqrt{\frac{1-\hat{s}_y(\mathbf{x})}{\hat{s}_y(\mathbf{x})}} \\
&= \tilde{h}(\mathbf{x})
\end{aligned}$$

in the support of q with the condition $\mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right] < \infty$ in (9) under Assumption 3. Therefore, the right-hand side of (S4.9) goes to zero as $n \to \infty$, so does the right-hand side of (S4.5), completing the proof of (S4.4).

Also, the numerator of the first term in (S4.3), $h_n(\mathbf{X}_k)$, converges in probability to $\tilde{h}(\mathbf{X}_k)$ as $n \to \infty$ because, for any $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}\Big(\Big|h_n(\mathbf{X}_k) - \tilde{h}(\mathbf{X}_k)\Big| > \epsilon\Big) &= \mathbb{P}\Big(\tilde{h}(\mathbf{X}_k) - h_n(\mathbf{X}_k) > \epsilon\Big) \\
&\le \frac{1}{\epsilon}\left(\mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right] - \mathbb{E}_q[h_n(\mathbf{X})]\right)
\end{aligned}$$

by Markov's inequality, and the right-hand side of the inequality goes to zero as $n \to \infty$ because $\mathbb{E}_q[h_n(\mathbf{X})] \to \mathbb{E}_q\left[\tilde{h}(\mathbf{X})\right]$ as $n \to \infty$ as shown above.

Thus, by the continuous mapping theorem applied to (S4.3), it follows that

$$\eta_k \xrightarrow{P} \tilde{\eta}_k. \tag{S4.10}$$

– Proof of $r(\eta_k) \xrightarrow{P} r(\tilde{\eta}_k)$: By definition, we prove the following convergence for any $\epsilon > 0$,

$$\mathbb{P}\left(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon\right) \to 0 \tag{S4.11}$$

as $m \to \infty$.
For any fixed $\delta > 0$, the left-hand side of (S4.11) can be expressed as

$$\begin{aligned}\mathbb{P}\left(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon\right) &= \mathbb{P}\left(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon, |\eta_k - \tilde{\eta}_k| > \delta\right) + \mathbb{P}\left(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon, |\eta_k - \tilde{\eta}_k| \le \delta\right) \\ &\equiv \alpha_1 + \alpha_2,\end{aligned}$$

where

$$\alpha_1 = \mathbb{P}\left(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon, |\eta_k - \tilde{\eta}_k| > \delta\right) \le \mathbb{P}\left(|\eta_k - \tilde{\eta}_k| > \delta\right) \to 0 \quad \text{(S4.12)}$$

as $m \to \infty$, because of (S4.10). On the other hand, to prove $\alpha_2 \to 0$, we define the set

$$G_\delta \equiv \{x \in \mathbb{R} - \mathcal{N} \mid \exists y : |r(y) - r(x)| > \epsilon, |y - x| \le \delta\}$$

for each $\delta > 0$. Because $r(x)$ in (S4.2) is continuous at $x \in \mathbb{R} - \mathcal{N}$, it follows that

$$\lim_{\delta \to 0} G_\delta = \emptyset,$$

which implies that $\mathbb{P}(\tilde{\eta}_k \in G_\delta) \to 0$ as $\delta \to 0$. Thus,

$$\begin{aligned}\mathbb{P}(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon, |\eta_k - \tilde{\eta}_k| \le \delta, \tilde{\eta}_k \notin \mathcal{N}) &\le \mathbb{P}(\tilde{\eta}_k \in G_\delta) \quad &\text{(S4.13)} \\ &\to 0 \quad &\text{(S4.14)}\end{aligned}$$

as $\delta \to 0$. Therefore,

$$\begin{aligned}\alpha_2 &= \mathbb{P}\left(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon, |\eta_k - \tilde{\eta}_k| \le \delta\right) \\ &= \mathbb{P}(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon, |\eta_k - \tilde{\eta}_k| \le \delta, \tilde{\eta}_k \notin \mathcal{N}) + \mathbb{P}(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon, |\eta_k - \tilde{\eta}_k| \le \delta, \tilde{\eta}_k \in \mathcal{N}) \\ &\le \mathbb{P}(\tilde{\eta}_k \in G_\delta) + \mathbb{P}(|r(\eta_k) - r(\tilde{\eta}_k)| > \epsilon, |\eta_k - \tilde{\eta}_k| \le \delta, \tilde{\eta}_k \in \mathcal{N}) \quad &\text{(S4.15)} \\ &\le \mathbb{P}(\tilde{\eta}_k \in G_\delta) + \mathbb{P}(\tilde{\eta}_k \in \mathcal{N}) \\ &= \mathbb{P}(\tilde{\eta}_k \in G_\delta) \quad &\text{(S4.16)} \\ &\to 0, \quad &\text{(S4.17)}\end{aligned}$$

as $\delta \to 0$. The inequality in (S4.15) is due to (S4.13). The equation in (S4.16) is due to the condition in (10). The convergence in (S4.17) is due to (S4.14).
In summary, (S4.12) and (S4.17) together imply (S4.11), completing the proof of $r(\eta_k) \xrightarrow{P} r(\tilde{\eta}_k)$ in (13). □

S5 Proof of Lemma 2

To prove $\hat{\sigma}_y^2 \xrightarrow{P} \sigma_y^2$ in (17), we want to show

$$\mathbb{P}\left(\left|\hat{\sigma}_y^2 - \sigma_y^2\right| > \epsilon\right) \to 0 \quad \text{(S5.1)}$$

for any $\epsilon > 0$, as $m \to \infty$. We bound the left-hand side of (S5.1) from above as follows:

$$\begin{aligned}\mathbb{P}\left(\left|\hat{\sigma}_y^2 - \sigma_y^2\right| > \epsilon\right) &= \mathbb{P}\left(\left|\hat{\sigma}_y^2 - \tilde{\sigma}_y^2 + \tilde{\sigma}_y^2 - \sigma_y^2\right| > \epsilon\right) \\ &\le \mathbb{P}\left(\left|\hat{\sigma}_y^2 - \tilde{\sigma}_y^2\right| + \left|\tilde{\sigma}_y^2 - \sigma_y^2\right| > \epsilon\right) \\ &\le \mathbb{P}\left(\left|\hat{\sigma}_y^2 - \tilde{\sigma}_y^2\right| > \epsilon/2\right) + \mathbb{P}\left(\left|\tilde{\sigma}_y^2 - \sigma_y^2\right| > \epsilon/2\right), \end{aligned} \tag{S5.2}$$

where

$$\tilde{\sigma}_y^2 \equiv \frac{1}{m-1} \sum_{i=1}^{m} \left(\frac{1}{\tilde{N}_i} \sum_{j=1}^{\tilde{N}_i} \mathbb{I}\left(Y_j^{(i)} > y\right) L_i - \hat{P}_n(y) \right)^2 .$$

To prove (S5.1), we show that the two terms in (S5.2) converge to zeros as follows.

- Proof of $\mathbb{P}\left(\left|\hat{\sigma}_y^2 - \tilde{\sigma}_y^2\right| > \epsilon/2\right) \to 0$: To simplify $\left|\hat{\sigma}_y^2 - \tilde{\sigma}_y^2\right|$, we first define

$$\bar{s}_y(\mathbf{X}_i) \equiv \frac{1}{N_i} \sum_{j=1}^{N_i} \mathbb{I}\left(Y_j^{(i)} > y\right)$$

and

$$\tilde{s}_y(\mathbf{X}_i) \equiv \frac{1}{\tilde{N}_i} \sum_{j=1}^{\tilde{N}_i} \mathbb{I}\left(Y_j^{(i)} > y\right).$$

Also, we simplify $\hat{\sigma}_y^2$ by algebraic operations as follows:

$$\begin{aligned}\hat{\sigma}_y^2 &= \frac{1}{m-1} \sum_{i=1}^{m} \left(\bar{s}_y(\mathbf{X}_i) L_i - \hat{P}_n(y)\right)^2 \\ &= \frac{1}{m-1} \sum_{i=1}^{m} \left(\bar{s}_y(\mathbf{X}_i)^2 L_i^2 - 2\bar{s}_y(\mathbf{X}_i) L_i \hat{P}_n(y) + \hat{P}_n^2(y)\right) \\ &= \frac{m}{m-1} \left(\frac{1}{m} \sum_{i=1}^{m} \bar{s}_y(\mathbf{X}_i)^2 L_i^2 - 2\hat{P}_n^2(y) + \hat{P}_n^2(y)\right) \\ &= \frac{m}{m-1} \left(\frac{1}{m} \sum_{i=1}^{m} \bar{s}_y(\mathbf{X}_i)^2 L_i^2 - \hat{P}_n^2(y)\right). \end{aligned}$$

Similarly, we can simplify $\tilde{\sigma}_y^2$ as

$$\tilde{\sigma}_y^2 = \frac{m}{m-1} \left(\frac{1}{m} \sum_{i=1}^{m} \tilde{s}_y(\mathbf{X}_i)^2 L_i^2 - \hat{P}_n^2(y)\right).$$

Then, we obtain

$$
\begin{aligned}
&\mathbb{P}\left(\left|\hat{\sigma}_y^2 - \tilde{\sigma}_y^2\right| > \epsilon/2\right) \\
&= \mathbb{P}\left(\left|\frac{m}{m-1}\left(\frac{1}{m}\sum_{i=1}^{m}\bar{s}_y(\mathbf{X}_i)^2 L_i^2 - \hat{P}_n^2(y)\right) - \frac{m}{m-1}\left(\frac{1}{m}\sum_{i=1}^{m}\tilde{s}_y(\mathbf{X}_i)^2 L_i^2 - \hat{P}_n^2(y)\right)\right| > \epsilon/2\right) \\
&= \mathbb{P}\left(\left|\frac{1}{m-1}\sum_{i=1}^{m}\left(\bar{s}_y(\mathbf{X}_i)^2 - \tilde{s}_y(\mathbf{X}_i)^2\right) L_i^2\right| > \epsilon/2\right) \\
&\le \frac{2}{\epsilon}\mathbb{E}_q\left[\left|\frac{1}{m-1}\sum_{i=1}^{m}\left(\bar{s}_y(\mathbf{X}_i)^2 - \tilde{s}_y(\mathbf{X}_i)^2\right) L_i^2\right|\right] && \text{(S5.3)} \\
&\le \frac{2}{\epsilon}\mathbb{E}_q\left[\frac{1}{m-1}\sum_{i=1}^{m}\left|\bar{s}_y(\mathbf{X}_i)^2 - \tilde{s}_y(\mathbf{X}_i)^2\right| L_i^2\right] \\
&= \frac{2}{\epsilon}\frac{1}{m-1}\sum_{i=1}^{m}\mathbb{E}_q\left[\left|\bar{s}_y(\mathbf{X}_i)^2 - \tilde{s}_y(\mathbf{X}_i)^2\right| L_i^2\right] \\
&= \frac{2}{\epsilon}\frac{m}{m-1}\mathbb{E}_q\left[\left|\bar{s}_y(\mathbf{X}_1)^2 - \tilde{s}_y(\mathbf{X}_1)^2\right| L_1^2\right] && \text{(S5.4)} \\
&= \frac{2}{\epsilon}\frac{m}{m-1}\mathbb{E}_q\left[\left|\bar{s}_y(\mathbf{X}_1) - \tilde{s}_y(\mathbf{X}_1)\right|\left(\bar{s}_y(\mathbf{X}_1) + \tilde{s}_y(\mathbf{X}_1)\right) L_1^2\right] \\
&\le \frac{2}{\epsilon}\frac{m}{m-1}\sqrt{\mathbb{E}_q\left[\left(\bar{s}_y(\mathbf{X}_1) - \tilde{s}_y(\mathbf{X}_1)\right)^2 L_1^2\right]}\sqrt{\mathbb{E}_q\left[\left(\bar{s}_y(\mathbf{X}_1) + \tilde{s}_y(\mathbf{X}_1)\right)^2 L_1^2\right]} && \text{(S5.5)} \\
&= \frac{2}{\epsilon}\frac{m}{m-1}\sqrt{\mathbb{E}_q\left[s_y(\mathbf{X}_1)\left(1 - s_y(\mathbf{X}_1)\right)\left|\frac{1}{N_1} - \frac{1}{\tilde{N}_1}\right| L_1^2\right]}\sqrt{\mathbb{E}_q\left[\left(\bar{s}_y(\mathbf{X}_1) + \tilde{s}_y(\mathbf{X}_1)\right)^2 L_1^2\right]}, && \text{(S5.6)} \\
&\to 0 && \text{(S5.7)}
\end{aligned}
$$

where the inequality in (S5.3) holds by Markov's inequality. The equality in (S5.4) holds because $\left|\bar{s}_y(\mathbf{X}_i)^2 - \tilde{s}_y(\mathbf{X}_i)^2\right| L_i^2, i = 1, \dots, m$ are identically distributed. The inequality in (S5.5) holds by the Cauchy–Schwarz inequality. The equality in (S5.6) holds by (27). The convergence in (S5.7) holds by the following three facts:

- The ratio, $\frac{m}{m-1}$, in (S5.6) goes to one as $m \to \infty$.
- The first square-rooted expectation in (S5.6),

$$
\sqrt{\mathbb{E}_q\left[s_y(\mathbf{X}_1)\left(1 - s_y(\mathbf{X}_1)\right)\left|\frac{1}{N_1} - \frac{1}{\tilde{N}_1}\right| L_1^2\right]},
$$

  goes to zero as $m \to \infty$, as it was shown that (28) goes to zero as $m \to \infty$ based on Assumption 2, Lemma 1, and the dominated convergence theorem.

– The second square-rooted expectation in (S5.6) is finite:

$$
\begin{aligned}
&\sqrt{\mathbb{E}_q\left[\left(\bar{s}_y(\mathbf{X}_1)+\tilde{s}_y(\mathbf{X}_1)\right)^2 L_1^2\right]} \\
&= \sqrt{\mathbb{E}_q[(\bar{s}_y(\mathbf{X}_1)^2 + 2\bar{s}_y(\mathbf{X}_1)\tilde{s}_y(\mathbf{X}_1) + \tilde{s}_y(\mathbf{X}_1)^2)\, L_1^2]} \\
&\le \sqrt{\mathbb{E}_q[(\bar{s}_y(\mathbf{X}_1) + 2\tilde{s}_y(\mathbf{X}_1) + \tilde{s}_y(\mathbf{X}_1))\, L_1^2]} \qquad \text{(S5.8)} \\
&= \sqrt{\mathbb{E}_q[(s_y(\mathbf{X}_1) + 2s_y(\mathbf{X}_1) + s_y(\mathbf{X}_1))\, L_1^2]} \\
&= 2\sqrt{\mathbb{E}_q[s_y(\mathbf{X}_1)L_1^2]} \\
&< \infty, \qquad \text{(S5.9)}
\end{aligned}
$$

where the inequality in (S5.8) holds because of $0 \le \bar{s}_y(\mathbf{X}_1) \le 1$ and $0 \le \tilde{s}_y(\mathbf{X}_1) \le 1$. The inequality in (S5.9) holds by Assumption 2.

– Proof of $\mathbb{P}\big(\lvert\tilde{\sigma}_y^2 - \sigma_y^2\rvert > \epsilon/2\big) \to 0$: By definition, we want to show

$$\tilde{\sigma}_y^2 \xrightarrow{P} \sigma_y^2. \qquad \text{(S5.10)}$$

Because

$$\tilde{\sigma}_y^2 = \frac{m}{m-1}\left(\frac{1}{m}\sum_{i=1}^{m}\tilde{s}_y(\mathbf{X}_i)^2 L_i^2 - \hat{P}_n^2(y)\right)$$

and

$$\sigma_y^2 = \mathbb{E}_q\left[\frac{1}{\tilde{N}} s_y(\mathbf{X})\left(1 - s_y(\mathbf{X})\right) L^2\right] + \mathbb{E}_q\left[s_y(\mathbf{X})^2 L^2\right] - p_y^2,$$

the convergence in probability in (S5.10) follows if

$$\frac{1}{m}\sum_{i=1}^{m}\tilde{s}_y(\mathbf{X}_i)^2 L_i^2 \xrightarrow{P} \mathbb{E}_q\left[\frac{1}{\tilde{N}} s_y(\mathbf{X})\left(1 - s_y(\mathbf{X})\right) L^2\right] + \mathbb{E}_q\left[s_y(\mathbf{X})^2 L^2\right] \qquad \text{(S5.11)}$$

and

$$\hat{P}_n^2(y) \xrightarrow{P} p_y^2. \qquad \text{(S5.12)}$$

– Proof of the convergence in probability in (S5.11): This convergence holds by the weak law of large numbers because $\tilde{s}_y(\mathbf{X}_i)^2 L_i^2, i = 1, \ldots, m$ are i.i.d. and

$$
\begin{aligned}
\mathbb{E}_q\left[\tilde{s}_y(\mathbf{X})^2 L^2\right] &= \mathbb{E}_q\left[\frac{1}{\tilde{N}} s_y(\mathbf{X})\left(1 - s_y(\mathbf{X})\right) L^2\right] + \mathbb{E}_q\left[s_y(\mathbf{X})^2 L^2\right] \qquad \text{(S5.13)} \\
&< \infty, \qquad \text{(S5.14)}
\end{aligned}
$$

where the equation in (S5.13) is derived in (32). The inequality in (S5.14) holds by Assumption 2 based on (33) and (34).

- Proof of the convergence in probability in (S5.12): We want to show

$$\mathbb{P}\Big(\Big|\hat{P}_n^2(y) - p_y^2\Big| > \epsilon'\Big) \to 0$$

for any $\epsilon' > 0$ as $m \to \infty$. Note that

$$\begin{aligned}\mathbb{P}\Big(\Big|\hat{P}_n^2(y) - p_y^2\Big| > \epsilon'\Big) &= \mathbb{P}\Big(\Big|\Big(\hat{P}_n(y) - p_y\Big)\Big(\hat{P}_n(y) + p_y\Big)\Big| > \epsilon'\Big)\\ &\le \mathbb{P}\Big(2\Big|\hat{P}_n(y) - p_y\Big| > \epsilon'\Big)\\ &= \mathbb{P}\Big(\Big|\hat{P}_n(y) - \tilde{P}_n(y) + \tilde{P}_n(y) - p_y\Big| > \epsilon'/2\Big)\\ &\le \mathbb{P}\Big(\Big|\hat{P}_n(y) - \tilde{P}_n(y)\Big| > \epsilon'/4\Big) + \mathbb{P}\Big(\Big|\tilde{P}_n(y) - p_y\Big| > \epsilon'/4\Big),\end{aligned}$$

where the right-hand side of the last inequality goes to zero because the first term,

$$\mathbb{P}\Big(\Big|\hat{P}_n(y) - \tilde{P}_n(y)\Big| > \epsilon'/4\Big) \to 0$$

as $m \to \infty$ by (24) and the second term,

$$\mathbb{P}\Big(\Big|\tilde{P}_n(y) - p_y\Big| > \epsilon'/4\Big) \to 0$$

as $m \to \infty$ by the weak law of large numbers because $\tilde{P}_n(y)$ is a sample mean of i.i.d. random variables with the finite mean of p_y as shown in (31) based on Assumption 1.

Because (S5.11) and (S5.12) hold, the convergence in probability in (S5.10) holds.

By (S5.7) and (S5.10), the right-hand side of the inequality in (S5.2) goes to zero, completing the proof of (S5.1) and, equivalently, (17).

□

S6 Proof of Corollary 1

Among the conditions in Theorem 1, Assumptions 1 and 2 directly depend on y. We show that the conditions in Assumptions 1 and 2 hold when y is replaced by $\tilde{y}$ for $\tilde{y} > y$. Then, it follows that Theorem 1 where y is replaced by $\tilde{y}$ holds.

- Assumption 1 with $\tilde{y}$ in place of y: If we replace y in Assumption 1 with $\tilde{y}$, the condition still holds because if $q(\mathbf{x}) = 0$, then

$$\begin{aligned}0 &\le \mathbb{P}(Y > \tilde{y} \mid \mathbf{X} = \mathbf{x})\, f(\mathbf{x})\\ &\le \mathbb{P}(Y > y \mid \mathbf{X} = \mathbf{x})\, f(\mathbf{x})\\ &= 0\end{aligned}$$

  for any $\mathbf{x}$.
- Assumption 2 with $\tilde{y}$ in place of y: If we substitute $\tilde{y}$ for y in Assumption 2, the condition remains satisfied because $\mathbb{E}_q\left[\mathbb{I}(Y > \tilde{y})L^2\right] \le \mathbb{E}_q\left[\mathbb{I}(Y > y)L^2\right] < \infty$ for $\tilde{y} > y$.

Therefore, it follows that Theorem 1 with $\tilde{y}$ in place of y holds for $\tilde{y} > y$. That is, $\mathbb{P}\Big(p_{\tilde{y}} \in \Big(\hat{P}_n(\tilde{y}) \pm z_{\alpha/2}\hat{\sigma}_{\tilde{y}}/\sqrt{m}\Big)\Big) \to 1 - \alpha$ holds for $\alpha \in (0, 1)$ as $m \to \infty$. □

S7 implementation Details on the NREL Simulators

The NREL wind turbine simulators, TurbSim (Jonkman, 2009) and FAST (Jonkman and Buhl Jr., 2005), are used in the case study in Section 5 of the paper. Given a 10-min average wind speed, **X**, as an input to TurbSim, it generates a 10-min stochastic wind profile. Given the wind profile, FAST produces the time series of structural and mechanical load responses of a wind turbine. Specifically, for TurbSim, we adopt the wind regime used in Moriarty (2008), which is the class I-B in the IEC Standard (International Electrotechnical Commission, 2005). For FAST, we use the NREL 5-MW baseline turbine model specification (Jonkman et al, 2009; Moriarty, 2008).

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