Reliability Evaluation of Large-Scale Systems With Identical Units

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***Abstract*—The reliability assessment of a large-scale system that considers its units' degradation is challenging due to the resulting dimensionality problem. We propose a new methodology that allows us to overcome difficulties in analyzing large-scale system dynamics, and devise analytical methods for finding the multivariate distribution of the dynamically changing system condition. When each unit's degradation condition can be classified into a finite number of states, and the transition distribution from one state to another is known, we obtain the asymptotic distribution of the number of units at each degradation state using fluid and diffusion limits. Specifically, we use a uniform acceleration technique, and obtain the time-varying mean vector and the covariance matrix of the number of units at multiple degradation states. When a state transition follows a non-Markovian deterioration process, we integrate phase-type distribution approximations with the fluid and diffusion limits. We show that, with any transition time distributions, the distribution of the number of units at multiple degradation conditions can be approximated by the multivariate Gaussian distribution as the total number of units gets large. The analytical results enable us to perform probabilistic assessment of the system condition during the system's service life. Our numerical studies suggest that the proposed methods can accurately characterize the stochastic evolution of the system condition over time.**

***Index Terms*—Degradation process, fluid and diffusion limits, Gaussian process, phase-type distribution, stochastic process.**

Acronyms and Abbreviations

RHS	right-hand side
ODE	ordinary differential equation
SDE	stochastic differential equation
EM	Expectation-Maximization

Notation

M	total number of degradation states
N	total number of units in the system, or system size
T	system's service life
λ_t^i	transition rate from State i to State $i+1$ at time t
$X^i(t)$	number of units at State i at time t among the total number units, N
$\mathbf{S}(t)$	$[X^1(t), X^2(t), \cdots, X^M(t)]'$, system status at time t
$Y_i(\cdot)$	standard Poisson process
η	accelerating factor (total number of units)
$\bar{x}^i(t)$	fluid limit of the number of units at State i at time t
$d^i(t)$	diffusion limit of the number of units at State i at time t
$\mathbf{D}(t)$	$[d^1(t), d^2(t), \cdots, d^M(t)]'$, vector of diffusion limits
$\mathbf{W}(t)$	multi-dimensional standard Brownian motion
$\mathbf{\Sigma}(t)$	covariance matrix of $\mathbf{D}(t)$
$X_j^i(t)$	number of units at State i and Phase j at time t
$p_j^i(t)$	transition probability from Phase j to Phase $j+1$ at State i
n_i	number of phases at State ith
$\lambda_j^i(t)$	rate of leaving Phase j of State i

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I. Introduction

THIS study proposes a new approach for evaluating the reliability of large-scale systems with units that follow the same stochastic degradation process. Examples of large-scale systems include a utility-scale wind farm with dozens or hundreds of wind turbines (here, each turbine or turbine component is a unit, and a wind farm is a system), or a solar park consisting of multiple solar panels. The global trend of constructing large-scale facilities underscores the need to improve system-level reliability assessments for cost-effective decision-making.

With technology advancements, abundant data become available for characterizing units' degradation processes [1]. In response, degradation-based reliability analysis has received significant attention in many applications. Extensive studies have been performed for modeling a probabilistic degradation process, and estimating a lifetime distribution of a single unit-system based on a Wiener process [2], [3], a Gamma process [4], an Inverse Gaussian process [5], and other stochastic processes [6], [7].

Also relevant to this study is the rich body of literature on reliability modeling and analysis of a complex system that consists of multiple components. Significant efforts have been made to characterize the reliability of the k-out-of-n system [8]–[10]. For example, among different types of standby redundancy (e.g., hot, warm, or cold standby), Amari *et al.* [10] recently investigated the reliability characteristics of systems

with warm standby redundancy considering the stochastic dependencies among component failure times. Some of the literature evaluates the lifetime distribution of a complex system with components that have different functions. The reliability of serial-parallel systems is also of interest, e.g., [11]–[13]. Anastasiadis *et al.* [14] study a multi-component system where random shocks cause damage in several components, and model the statistical dependence among the components. In these studies that model the complex multi-component systems, analytical methods are derived for simplified models with several key assumptions, such as multiplicative aging structure and Poisson shock processes [14], whereas simulations have been often utilized for evaluating the reliability of a system with a high degree of complexity [15].

A. Problem Description

Reliability studies in the literature primarily focus on estimating the time-to-failure distribution of a system. However, when a system consists of multiple units, its degradation status can inform maintenance decisions. Our study aims to extract system-level information from its units' degradation process information. When a system consists of one or a small number of units, the analysis of the system condition would be tractable. However, when a large number of units operate in a system, translating the degradation condition of individual units into system-level information remains a significant challenge. We develop a general procedure to characterize the stochastic evolution of the number of units at multiple degradation conditions over a system's service life.

The degradation condition in many engineering units can be modeled as a continuous gradual process [16]. Continuous degradation processes, however, can be translated into discrete multi-state processes via suitable discretization techniques depending on the research context. Multi-state conditions have been used in many studies to represent the different levels of system performance or degradation levels (e.g., normal, alert, alarm and failure states) [12], [13], [17], [18]. Fig. 1 illustrates an example of such multi-state conditions. In this study, we classify a unit's degradation condition into a finite number of states for computational tractability.

We assume that the distribution for the transition time from one state to the next deteriorated state is known (or estimated from historical data). It is also assumed that each unit s-independently degrades from State i to State $i+1$, $i = 1, \cdots, M-1$, following the same stochastic degradation process. Here, State 1 denotes the best condition, State $M-1$ reflects the most deteriorated condition, and State M is the failed condition [17]. All of the units start from a normal state in the beginning of operations, i.e., $X^1(0) = N$ and $X^i(0) = 0$ for $i = 2, \ldots, M$, but their degradation stochastically progresses. In this study, we define the system status as a vector of the number of units in the system at each degradation state. Therefore, the system status at time t, $\mathbf{S}(t)$, is an M-dimensional stochastic process, defined as

$$\mathbf{S}(t) = \left(X^1(t), \ldots, X^M(t)\right)'.$$

The objective of this study is to derive the asymptotic multivariate distribution of the dynamically changing system status during the system's service life, $[0, T]$, when the number of units in the system is large. Notice that the system reliability concerned in this study differs from the general notion of the probability that a system operates (e.g., probability that a unit's degradation level is less than a specified threshold level). Rather, we focus on obtaining the degradation information at the system level, and tracking the change of the system status during operations.

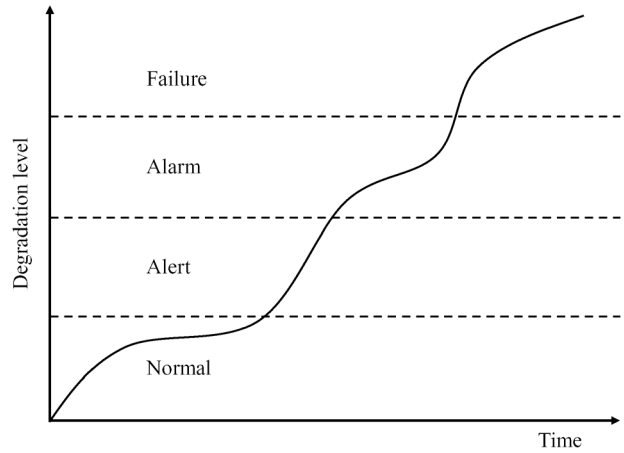

Fig. 1. Example of discretizing a continuous degradation process into a finite number of states.

B. Summary of Research Results and Contributions

We consider a system where each unit's degradation transition time follows any general distribution. We first model the stochastic process where each unit degrades with a Markovian process (i.e., the transition from one state to another follows a Poisson process). The Poisson process can be either homogeneous with fixed parameters, or non-homogeneous with time-varying or system status-dependent parameters. Then, we relax the Markovian assumption, and allow a general non-Markovian transition process. It is well known that relaxing the Markovian assumption causes significant difficulties in analyzing the dynamic characteristics of a system. We devise a novel method that makes the analysis of a large-scale non-Markovian system tractable. Specifically, we employ a phase-type distribution to approximate general distributions, and extend the techniques developed for the Markovian process.

To characterize the large-scale system dynamics, we employ the uniform acceleration technique. The uniform acceleration technique has been used in analyzing large-scale queueing systems and networks [19]–[21]. The technique's general applicability for many other stochastic systems can also be found at [22] and [23]. The basic idea is to obtain fluid and diffusion limits by accelerating some relevant parameters (e.g., the number of units in this study) to infinity while suitably adjusting other parameters. Knowing the two limits allows us to find the evolution of both the average and the variance (or covariance matrix) of the system status. Our numerical studies, using data available in the literature, show that the proposed methods can accurately capture the probabilistic progression of the system status.

We emphasize the main contributions with the following points. First, we devise a new analytical and computational

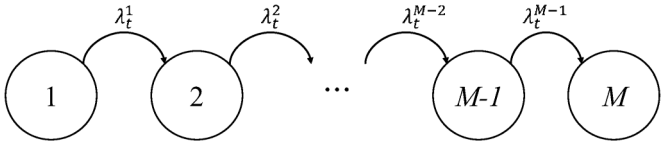

Fig. 2. Markovian degradation process.

approach to characterize the system status when units degrade according to a Markovian or non-Markovian process. To the best of our knowledge, the presented model is the first mathematical model for the reliability analysis of large-sale systems considering both Markovian and non-Markovian degradation processes. Consequently, the presented approach is generally applicable to a system where its units' transition time follows any distribution. Secondly, the proposed approach is computationally tractable. In the past, analysis of the system dynamics with many units, and usage of phase-type distributions with many phases, have been limited to small-scale problems due to the curse of dimensionality. The presented method, however, achieves better scalability by combining limit processes and phase-type distributions. In our numerical study, we use 10 phases per each state for better accuracy, which is unusual in the previous studies that use only a few phases [24], [25]. We also obtain numerical solutions within a few seconds, even for the case of 4 states and 10 phases per each state (Section IV-B). Lastly, we show that the probability distribution of the system status asymptotically converges to the multivariate Gaussian distribution with a time-varying mean and a covariance matrix whose values depend on the system's age. These results allow us to perform probabilistic assessment of the system condition during the system's service life, and provide rich information useful for operations and maintenance decision-making [16]–[18], [26]. For instance, we can track how many units will be in a critically degraded state (e.g., alarm condition) over time. One can use such information for warehouse sizing for spare parts, and maintenance resource planning for ensuring reliable system operations.

In the remainder of the paper, Section II considers a Markovian degradation process, and Section III extends the approach for a non-Markovian degradation process. Section IV provides numerical results. Finally, we conclude the paper in Section V.

II. System Status Assessment With Markovian Degradation Process

This section develops a method for evaluating the system status when each unit's degradation process has a memoryless property, and thus follows a Markovian degradation process (see Fig. 2). The presented method in this section will be used as a basis for evaluating the system status in a general, non-Markovian degradation process in Section III.

A. Model

In the Markovian degradation process, each unit starts from State 1 at time 0, moves to the next state after a random amount of time, and continues until it reaches the failure state, M. The number of units at State i at time t, $X^i(t)$, $i = 1, \cdots, M$, can be obtained by solving the following integral equations.

$$X^1(t) = N - Y_1\left(\int_0^t \lambda_s^1 X^1(s)ds\right), \tag{1}$$

$$X^2(t) = Y_1\left(\int_0^t \lambda_s^1 X^1(s)ds\right) - Y_2\left(\int_0^t \lambda_s^2 X^2(s)ds\right), \tag{2}$$

$$X^3(t) = Y_2\left(\int_0^t \lambda_s^2 X^2(s)ds\right) - Y_3\left(\int_0^t \lambda_s^3 X^3(s)ds\right), \tag{3}$$

$$\vdots$$

$$X^{M-1}(t) = Y_{M-2}\left(\int_0^t \lambda_s^{M-2} X^{M-2}(s)ds\right) - Y_{M-1}\left(\int_0^t \lambda_s^{M-1} X^{M-1}(s)ds\right), \tag{4}$$

$$X^M(t) = Y_{M-1}\left(\int_0^t \lambda_s^{M-1} X^{M-1}(s)ds\right), \tag{5}$$

where Y_i, $i = 1, \cdots, M-1$, is a mutually s-independent standard Poisson process [22], denoting the number of arrivals or departures up to time t, depending on the sign before it. The integrand in $Y_i(\cdot)$ is the rate of arrival or departure at time s, which is, in fact, the rate of a non-homogeneous Poisson process. In (1)–(4), the two terms on the right-hand side (RHS) represent the counting processes of inputs to the corresponding state, and outputs from the state, respectively. Therefore, $X^i(t)$, $i = 1 \cdots, M$, can be obtained by counting the number of arrivals to State i, and departures from State i, until time t. In (1), the number of units at State 1, which starts from N at time 0, decreases by $Y_1(\cdot)$ because $Y_1(\cdot)$ counts the number of departures from State 1. Note that, in (2)–(4), the first term on the RHS, the input to the corresponding state, is the same as the number of departures from the previous state. Because $X^M(t)$ in (5) is the failure state, it only includes the input counting process.

For general applicability, we use time-varying transition rates (see the subscript t of λ_t^i, $i = 1, \cdots, M$, in (1)–(5), and Fig. 2). Using the time-varying rates enables us to incorporate temporal variations, such as daily or seasonal effects on the degradation process. When a constant transition rate is used (i.e., $\lambda_t^i = \lambda^i$), the transition time follows the exponential distribution. With the time-varying rates, the transition time is not exponentially distributed, having a different distribution (see [27] for the details).

To solve (1)–(5), the Matrix Geometric method, a well-known analytic methodology for Markov Chains [28], can be used for small-scale systems. However, when a system consists of a large number of units, it suffers from poor scalability because the problem size increases on the order of M^N. In the next section, we apply the uniform acceleration technique [19]–[21], and find an approximate solution by using fluid and diffusion limits for large-scale systems.

B. Fluid and Diffusion Limits

We introduce an accelerating factor, η, to apply the uniform acceleration technique. The accelerating factor is indeed the number of units, N, in the system. We increase the accelerating factor (i.e., the number of units) to infinity, and adjust the terms inside $Y_i(\cdot)$ in (1)–(5) accordingly. We rewrite (1)–(5), and consider a set of the following stochastic processes, $\{X^{i,\eta}(t)\}_{\eta\geq 1}$.

$$\begin{aligned}
X^{1,\eta}(t) &= \eta - Y_1\left(\int_0^t \eta\lambda_s^1 \frac{X^{1,\eta}(s)}{\eta}ds\right),\\
X^{2,\eta}(t) &= Y_1\left(\int_0^t \eta\lambda_s^1 \frac{X^{1,\eta}(s)}{\eta}ds\right)\\
&\quad - Y_2\left(\int_0^t \eta\lambda_s^2 \frac{X^{2,\eta}(s)}{\eta}ds\right),\\
X^{3,\eta}(t) &= Y_2\left(\int_0^t \eta\lambda_s^2 \frac{X^{2,\eta}(s)}{\eta}ds\right)\\
&\quad - Y_3\left(\int_0^t \eta\lambda_s^3 \frac{X^{3,\eta}(s)}{\eta}ds\right),\\
&\vdots\\
X^{M-1,\eta}(t) &= Y_{M-2}\left(\int_0^t \eta\lambda_s^{M-2} \frac{X^{M-2,\eta}(s)}{\eta}ds\right)\\
&\quad - Y_{M-1}\left(\int_0^t \eta\lambda_s^{M-1} \frac{X^{M-1,\eta}(s)}{\eta}ds\right)\\
X^{M,\eta}(t) &= Y_{M-1}\left(\int_0^t \eta\lambda_s^{M-1} \frac{X^{M-1,\eta}(s)}{\eta}ds\right).
\end{aligned}$$

Then, by taking $\eta\to\infty$, we obtain the fluid limit in Proposition 1. (Detailed proofs are included in the Appendix.)

Proposition 1: Suppose that T is a service life of the system. On $t\in[0,T]$ for $T>0$,

$$\lim_{\eta\to\infty}\frac{X^{i,\eta}(t)}{\eta} = \bar{x}^i(t)\ almost\ surely,$$

where $\bar{x}^1(0)=1$, $\bar{x}^i(0)=0$ for $i\in\{2,\ldots,M\}$, and

$$\begin{aligned}
\frac{d}{dt}\bar{x}^1(t) &= -\lambda_t^1\bar{x}^1(t),\\
\frac{d}{dt}\bar{x}^2(t) &= \lambda_t^1\bar{x}^1(t) - \lambda_t^2\bar{x}^2(t),\\
\frac{d}{dt}\bar{x}^3(t) &= \lambda_t^2\bar{x}^2(t) - \lambda_t^3\bar{x}^3(t),\\
&\vdots\\
\frac{d}{dt}\bar{x}^{M-1}(t) &= \lambda_t^{M-2}\bar{x}^{M-2}(t) - \lambda_t^{M-1}\bar{x}^{M-1}(t),\\
\frac{d}{dt}\bar{x}^{M}(t) &= \lambda_t^{M-1}\bar{x}^{M-1}(t).
\end{aligned}$$

The fluid limit, $\bar{x}^i(t)$, $i=1,\cdots,M$, represents the average fraction of units at State i at time t among N units. This fluid limit will be used to find the approximate mean value of the number of units at State i (see Lemma 1 and the following discussions). The following remark states that the sum of the fluid limits is equal to one.

Remark 1: On $t\in[0,T]$ for $T>0$,

$$\sum_{i=1}^{M}\bar{x}^i(t) = 1.$$

Next, to obtain the diffusion limit, we define two matrices. Define an $M\times M$ matrix, $\mathbf{A}(t)=(a_{i,j}(t))$, as follows.

$$\begin{aligned}
a_{i,i}(t) &= -\lambda_t^i \text{ for } 1\leq i\leq M-1,\\
a_{i,i-1}(t) &= \lambda_t^{i-1} \text{ for } 2\leq i\leq M.
\end{aligned}$$

Define another $M\times M$ matrix, $\mathbf{B}(t)=(b_{i,j}(t))$, as follows.

$$\begin{aligned}
b_{i,i}(t) &= -\sqrt{\lambda_t^i\bar{x}^i(t)} \text{ for } 1\leq i\leq M-1,\\
b_{i,i-1}(t) &= \sqrt{\lambda_t^{i-1}\bar{x}^{i-1}(t)} \text{ for } 2\leq i\leq M.
\end{aligned}$$

Now, with the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$, Proposition 2 presents the diffusion limit.

Proposition 2: On $t\in[0,T]$ for $T>0$,

$$\lim_{\eta\to\infty}\sqrt{\eta}\left\{\frac{X^{i,\eta}(t)}{\eta} - \bar{x}^i(t)\right\} = D^i(t)\ in\ distribution,$$

for $i=1,\cdots,M$, where $\mathbf{D}(t)=(D^1(t),\ldots,D^M(t))'$ is the solution to the following stochastic differential equation (SDE).

$$d\mathbf{D}(t) = \mathbf{A}(t)\mathbf{D}(t)dt + \mathbf{B}(t)d\mathbf{W}(t), \mathbf{D}(0)=0, \qquad (6)$$

and $\mathbf{W}(t)$ is an M-dimensional standard Brownian motion. The covariance matrix of $\mathbf{D}(t)$, $\mathbf{\Sigma}(t)$, is the solution to the following ordinary differential equations (ODEs).

$$\frac{d}{dt}\mathbf{\Sigma}(t) = \mathbf{A}(t)\mathbf{\Sigma}(t) + \mathbf{\Sigma}(t)\mathbf{A}(t)' + \mathbf{B}(t)\mathbf{B}(t)'. \qquad (7)$$

With the obtained fluid and diffusion limits, Lemma 1 states that the diffusion limits follow the multivariate Gaussian distribution.

Lemma 1: The diffusion limit process, $\mathbf{D}(t)$, is a Gaussian process. That is, for $t\in[0,T]$, $\mathbf{D}(t)=(D^1(t),\ldots,D^M(t))$ follows the multivariate normal distribution with the mean being an $M\times 1$ zero vector, and the covariance matrix $\mathbf{\Sigma}(t)$.

The results in Propositions 1 and 2, and Lemma 1, lead us to derive the asymptotic multivariate distribution of the system status. Using Proposition 2, for a large η, we can approximate $X^{i,\eta}(t)$ as follows.

$$X^{i,\eta}(t) \approx \eta\bar{x}^i(t) + \sqrt{\eta}D^i(t) \text{ in distribution},$$

and from Lemma 1, we obtain

$$\begin{aligned}
\mathrm{E}\left[X^{i,\eta}(t)\right] &\approx \eta\bar{x}^i(t),\\
\mathrm{Var}\left[X^{i,\eta}(t)\right] &\approx \eta\mathrm{Var}\left[D^i(t)\right],
\end{aligned}$$

and the covariance matrix of $\mathbf{S}(t)$ is approximated by $\eta\boldsymbol{\Sigma}(t)$. As such, for the system with N units, we can find the approximate mean and variance of the number of units at each state, $i = 1, \cdots, M$,

$$\begin{aligned} \mathrm{E}\left[X^i(t)\right] &\approx N\bar{x}^i(t), \\ \mathrm{Var}\left[X^i(t)\right] &\approx N\mathrm{Var}\left[D^i(t)\right], \end{aligned}$$

and the covariance matrix of $\mathbf{S}(t)$ is approximately given by $N\boldsymbol{\Sigma}(t)$. Therefore, the distribution of the system status at time t, $\mathbf{S}(t)$, can be approximated by the multivariate normal distribution with a mean vector $N[\bar{x}^1(t), \cdots, \bar{x}^M(t)]'$, and a covariance matrix, $N\boldsymbol{\Sigma}(t)$, i.e., $\mathbf{S}(t) \sim MVN(N[\bar{x}^1(t), \cdots, \bar{x}^M(t)]', N\boldsymbol{\Sigma}(t))$. Because $\bar{x}^i(t)$ denotes a fraction of the number of units at State i, the sum of the elements in the mean vector of the system status distribution is equal to the total number of units in the system, i.e., $\sum_{i=1}^{M} N\bar{x}^i(t) = N$.

C. Implications

Our proposed method with the Markovian degradation process provides some important benefits. First, the model discussed in Section II-A is intuitive. Recall that we formulate the model by counting inputs and outputs at each state. Also, our model allows the use of time-varying transition rates to incorporate heterogeneous degradation rates that can be caused by dynamic environmental and operations conditions such as seasonal variations.

Another benefit can be found at its computational efficiency for solving large-size problems, overcoming the computational limitations of existing techniques such as the Matrix Geometric method [28]. Even for a system with several hundred or thousand units, the ODEs in fluid and diffusion limits can be obtained within a few seconds using numerical solvers in a standard desktop computer. In fact, the computational complexity in the ODEs in Propositions 1 and 2 is not affected by the number of units because the accelerating factor, η, will disappear as $\eta \to \infty$.

In spite of the aforementioned benefits, we acknowledge that the assumption of the Markovian process, namely the memoryless property, could be restrictive in real applications. In the next section, we relax this assumption, and analyze a general non-Markovian system.

III. System Status Assessment With Non-Markovian Degradation Process

When the Markovian assumption is removed from the degradation process, tracking the dynamics of a large-scale system becomes extremely difficult. That is the main reason many studies have been focusing on the Markovian process in the past. In reality, the lifetime distribution of units can be any suitable distribution. In fact, Weibull, lognormal, and gamma distributions have been widely used to describe the lifetime of units [29]. This section extends the methodology presented in Section II, and considers a degradation model where the transition time between two states follows a general distribution.

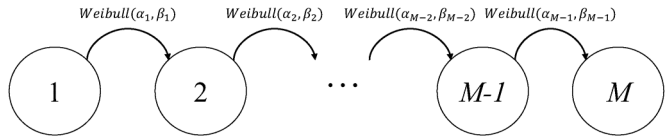

Fig. 3. Example of a non-Markovian degradation process.

For example, Fig. 3 depicts the degradation process where each transition time follows the Weibull distribution.

A. Approximation With Phase-Type Distributions

The fluid and diffusion limits developed in the Markovian degradation process in Section II cannot be directly applied to a non-Markovian process because all of the limit processes are obtained under the Markovian assumption. To evaluate the system's status in the non-Markovian degradation process, our main idea is to integrate fluid and diffusion limits with phase-type distributions. Phase-type distributions have been used to approximate other distributions, and it is well known that phase-type distributions are dense in all positive-support distributions [30]. Therefore, we approach the problem by approximating the transition distributions (e.g., $Weibull(\alpha_i, \beta_i)$, $i = 1, \cdots, M-1$, in Fig. 3) using phase-type distributions, and then combining the results with fluid and diffusion limits.

A phase-type distribution is a probability distribution constructed by a mixture of exponential distributions [28]. There are several well-known phase-type distributions including Erlang, hyper-exponential, and Coxian distributions; and many algorithms have been developed to approximate general distributions with phase-type distributions [30]–[34]. Among them, Asmussen *et al.* [30] show that the Coxian distribution produces high approximation quality for many distributions including Weibull and lognormal distributions. Therefore, we select the Coxian distribution, and use the Expectation-Maximization (EM) algorithm to estimate the Coxian distribution parameters [30]. The Coxian distriubtion can also provide a good fit when empirical transition time data are available. It, however, should be noted that our focus is not on developing an algorithm for finding phase-type distributions; we can employ any appropriate phase-type distribution suitable for given transition time distributions or data.

The Coxian distribution consists of several phases. For example, Fig. 4 illustrates a Coxian distribution with five phases for approximating the transition time density from State 1 to State 2, showing how the Coxian distribution works inside each state and between states. One can see that the Coxian distribution allows a transition from any phase at one state to the first phase at the next state. Suppose that there are n_i phases for the transition distribution from State i to $i+1$, $i = 1, \cdots, M-1$. Let λ_j^i and p_j^i be the parameters of the Coxian distribution for Phase j of State i (see Fig. 4), $i = 1, \cdots, M-1$, $j = 1, \cdots, n_i$. Given the original transition distribution (e.g., $Weibull(\alpha_i, \beta_i)$), we generate transition time data from the distribution, and estimate the Coxian distribution parameters, λ_j^i and p_j^i, using the EM algorithm [30] for $i = 1, \cdots, M-1$, and $j = 1, \cdots, n_i$.

In the past, in spite of the accurate approximation capability of the phase-type distributions, their usage has been limited to

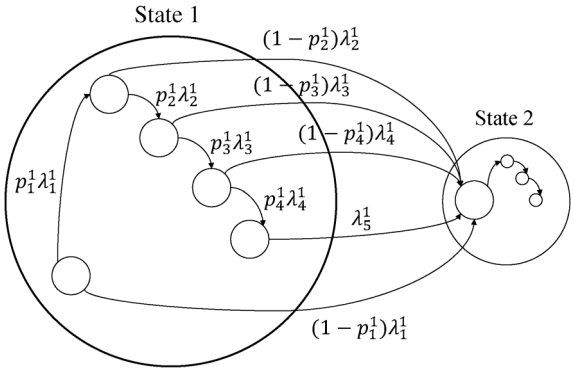

Fig. 4. Approximation of a transition distribution with the Coxian distribution.

small-scale problems due to their poor scalability. We overcome this limitation by employing fluid and diffusion limits in the next section.

B. Model

We use the results from the Coxian distribution approximation to model the system status. We define

$$\mathbf{X}(t) = \left(X_1^1(t), \ldots, X_{n_1}^1(t), \ldots X_1^{M-1}(t), \ldots, X_{n_{M-1}}^{M-1}(t), X_1^M(t)\right)',$$

each element of which is the number of units at the corresponding phase and state at time t. The initial value is $\mathbf{X}(0) = (N, 0, \ldots, 0)'$. Note that each phase is virtual, and only used for approximating the original distribution. We can model $\mathbf{X}(t)$ with the following integral equations.

$$\begin{aligned}
X_1^1(t) &= N - Y_{1,1}^1\left(\int_0^t p_1^1\lambda_1^1 X_1^1(s)ds\right) \\
&\quad - Y_{1,2}^1\left(\int_0^t \left(1-p_1^1\right)\lambda_1^1 X_1^1(s)ds\right), \\
X_1^i(t) &= \sum_{l=1}^{n_{i-1}-1} Y_{l,2}^{i-1}\left(\int_0^t \left(1-p_l^{i-1}\right)\lambda_l^{i-1}X_l^{i-1}(s)ds\right) \\
&\quad + Y_{n_{i-1},1}^{i-1}\left(\int_0^t \lambda_{n_{i-1}}^{i-1}X_{n_{i-1}}^{i-1}(s)ds\right) \\
&\quad - Y_{1,1}^i\left(\int_0^t p_1^i\lambda_1^i X_1^i(s)ds\right) \\
&\quad - Y_{1,2}^i\left(\int_0^t \left(1-p_1^i\right)\lambda_1^i X_1^i(s)ds\right) \text{ for } 2\le i\le M-1, \\
X_j^i(t) &= Y_{j-1,1}^i\left(\int_0^t p_{j-1}^i\lambda_{j-1}^i X_{j-1}^i(s)ds\right) \\
&\quad - Y_{j,1}^i\left(\int_0^t p_j^i\lambda_j^i X_j^i(s)ds\right) \\
&\quad - Y_{j,2}^i\left(\int_0^t \left(1-p_j^i\right)\lambda_j^i X_j^i(s)ds\right) \\
&\quad \text{for } 1\le i\le M-1, 2\le j\le n_i-1, \\
X_{n_i}^i(t) &= Y_{n_1-1,1}^i\left(\int_0^t p_{n_i-1}^i\lambda_{n_i-1}^i X_{n_i-1}^i(s)ds\right) \\
&\quad - Y_{n_i,1}^i\left(\int_0^t \lambda_{n_i}^i X_{n_i}^i(s)ds\right) \text{ for } 1\le i\le M-1, \\
X_1^M(t) &= \sum_{l=1}^{n_{M-1}-1} Y_{l,2}^{M-1}\left(\int_0^t \left(1-p_l^{M-1}\right)\lambda_l^{M-1}X_l^{M-1}(s)ds\right) \\
&\quad + Y_{n_{M-1},1}^{M-1}\left(\int_0^t \lambda_{n_{M-1}}^{M-1}X_{n_{M-1}}^{M-1}(s)ds\right).
\end{aligned}$$

Similar to the model in the Markovian degradation process in Section II-A, each term on the RHS can be interpreted using input and output counting processes. The difference is that input and output can be made among phases either inside a state or between states.

C. Fluid and Diffusion Limits

The model in Section III-B is not solvable for large-scale systems. We use the fluid and diffusion limits by accelerating the number of units. Consider a sequence of stochastic processes, $\{\mathbf{X}^\eta(t)\}_{\eta\ge 1}$,

$$\begin{aligned}
X_1^{1,\eta}(t) &= \eta - Y_{1,1}^1\left(\int_0^t \eta p_1^1\lambda_1^1\frac{X_1^{1,\eta}(s)}{\eta}ds\right) \\
&\quad - Y_{1,2}^1\left(\int_0^t \eta\left(1-p_1^1\right)\lambda_1^1\frac{X_1^{1,\eta}(s)}{\eta}ds\right), \\
X_1^{i,\eta}(t) &= \sum_{l=1}^{n_{i-1}-1} Y_{l,2}^{i-1}\left(\eta\int_0^t \left(1-p_l^{i-1}\right)\lambda_l^{i-1}\frac{X_l^{i-1,\eta}(s)}{\eta}ds\right) \\
&\quad + Y_{n_{i-1},1}^{i-1}\left(\int_0^t \eta\lambda_{n_{i-1}}^{i-1}\frac{X_{n_{i-1}}^{i-1,\eta}(s)}{\eta}ds\right), \\
&\quad - Y_{1,1}^i\left(\int_0^t \eta p_1^i\lambda_1^i\frac{X_1^{i,\eta}(s)}{\eta}ds\right) \\
&\quad - Y_{1,2}^i\left(\int_0^t \eta\left(1-p_1^i\right)\lambda_1^i\frac{X_1^{i,\eta}(s)}{\eta}ds\right) \\
&\quad \text{for } 2\le i\le M-1, \\
X_j^{i,\eta}(t) &= Y_{j-1,1}^i\left(\int_0^t \eta p_{j-1}^i\lambda_{j-1}^i\frac{X_{j-1}^{i,\eta}(s)}{\eta}ds\right) \\
&\quad - Y_{j,1}^i\left(\int_0^t \eta p_j^i\lambda_j^i\frac{X_j^{i,\eta}(s)}{\eta}ds\right)
\end{aligned}$$

$$- Y^i_{j,2}\left(\int_0^t \eta\left(1-p^i_j\right)\lambda^i_j\frac{X^{i,\eta}_j(s)}{\eta}ds\right)$$
$$\text{for } 1\le i\le M-1, 2\le j\le n_i-1,$$
$$X^{i,\eta}_{n_i}(t) = Y^i_{n_1-1,1}\left(\int_0^t \eta p^i_{n_i-1}\lambda^i_{n_i-1}\frac{X^{i,\eta}_{n_i-1}(s)}{\eta}ds\right)$$
$$- Y^i_{n_i,1}\left(\int_0^t \eta\lambda^i_{n_i}\frac{X^{i,\eta}_{n_i}(s)}{\eta}ds\right) \text{ for } 1\le i\le M-1,$$
$$X^{M,\eta}_1(t) = \sum_{l=1}^{n_{M-1}-1} Y^{M-1}_{l,2}$$
$$\times\left(\int_0^t \eta\left(1-p^{M-1}_l\right)\lambda^{M-1}_l\frac{X^{M-1,\eta}_l(s)}{\eta}ds\right)$$
$$+ Y^{M-1}_{n_{M-1},1}\left(\int_0^t \eta\lambda^{M-1}_{n_{M-1}}\frac{X^{M-1,\eta}_{n_{M-1}}(s)}{\eta}ds\right).$$

Then, the solution process to find the asymptotic distribution of the system status is similar to that in Section II. First, by taking $\eta\to\infty$, we derive the fluid limit, which leads to Proposition 3.

Proposition 3: On $t\in[0,T]$ for $T>0$,

$$\lim_{\eta\to\infty}\frac{X^{i,\eta}_j(t)}{\eta} = \bar{x}^i_j(t) \text{ almost surely},$$

where $\bar{x}^1_1(0)=1$, and $\bar{x}^i_j(0)=0$ for all i,j except $i=1$ and $j=1$; and

$$\frac{d}{dt}\bar{x}^1_1(t) = -\lambda^1_1\bar{x}^1_1(t),$$
$$\frac{d}{dt}\bar{x}^i_1(t) = \sum_{l=1}^{n_{i-1}-1}\left(1-p^{i-1}_l\right)\lambda^{i-1}_l\bar{x}^{i-1}_l(t)$$
$$+\lambda^{i-1}_{n_{i-1}}\bar{x}^{i-1}_{n_{i-1}}(t) - \lambda^i_1\bar{x}^i_1(t) \text{ for } 2\le i\le M-1,$$
$$\frac{d}{dt}\bar{x}^i_j(t) = p^i_{j-1}\lambda^i_{j-1}\bar{x}^i_{j-1}(t) - \lambda^i_j\bar{x}^i_j(t)$$
$$\text{for } 1\le i\le M-1, 2\le j\le n_i-1,$$
$$\frac{d}{dt}\bar{x}^i_{n_i}(t) = p^i_{n_i-1}\lambda^i_{n_i-1}\bar{x}^i_{n_i-1}(t) - \lambda^i_{n_i}\bar{x}^i_{n_i}(t)$$
$$\text{for } 1\le i\le M-1,$$
$$\frac{d}{dt}\bar{x}^M_1(t) = \sum_{l=1}^{n_{M-1}-1}\left(1-p^{M-1}_l\right)\lambda^{M-1}_l\bar{x}^{M-1}_l(t)$$
$$+\lambda^{M-1}_{n_{M-1}}\bar{x}^{M-1}_{n_{M-1}}(t).$$

The fluid limit, $\bar{x}^i_j(t)$, implies the fraction of the units at Phase j of State i. Therefore, the sum of all of the fluid limits becomes one as Remark 2 states.

Remark 2: On $t\in[0,T]$ for $T>0$,

$$\sum_{i=1}^{M}\sum_{j=1}^{n_i}\bar{x}^i_j(t) = 1.$$

Next, we define the following two matrices for deriving the diffusion limit. Define a $[(\sum_{i=1}^{M-1}n_i)+1]\times[(\sum_{i=1}^{M-1}n_i)+1]$ matrix $\mathbf{A}(t)=(a_{i,j}(t))$ whose (i,j)th element, for $1\le i\le M-1$ and $2\le j\le n_i$, is

$$a_{\left(\sum_{k=0}^{i-1}n_k\right)+1,\left(\sum_{k=0}^{i-1}n_k\right)+1}(t) = -\lambda^i_1,$$
$$a_{\left(\sum_{k=0}^{i-1}n_k\right)+j,\left(\sum_{k=0}^{i-1}n_k\right)+j-1}(t) = p^i_{j-1}\lambda^i_{j-1},$$
$$a_{\left(\sum_{k=0}^{i-1}n_k\right)+j,\left(\sum_{k=0}^{i-1}n_k\right)+j}(t) = -\lambda^i_j,$$
$$a_{\left(\sum_{k=0}^{i}n_k\right)+1,\left(\sum_{k=0}^{i-1}n_k\right)+j-1}(t) = \left(1-p^i_{j-1}\right)\lambda^i_{j-1},$$
$$a_{\left(\sum_{k=0}^{i}n_k\right)+1,\left(\sum_{k=0}^{i}n_k\right)}(t) = \lambda^i_{n_i},$$

where $n_0=0$. Also, define a $[(\sum_{i=1}^{M-1}n_i)+1]\times[\sum_{i=1}^{M-1}(2n_i-1)]$ matrix $\mathbf{B}(t)=(b_{i,j}(t))$ whose (i,j)th element is

$$b_{j,2j-1}(t) = -\sqrt{p^1_j\lambda^1_j\bar{x}^1_j},$$
$$b_{j,2j}(t) = -\sqrt{\left(1-p^1_j\right)\lambda^1_j\bar{x}^1_j},$$
$$b_{n_1+1,2j}(t) = \sqrt{\left(1-p^1_j\right)\lambda^1_j\bar{x}^1_j},$$
$$b_{j+1,2j-1}(t) = \sqrt{p^1_j\lambda^1_j\bar{x}^1_j},$$
$$b_{n_1,2n_1-1}(t) = -\sqrt{\lambda^1_{n_1}\bar{x}^1_{n_1}},$$
$$b_{n_1+1,2n_1-1}(t) = \sqrt{\lambda^1_{n_1}\bar{x}^1_{n_1}},$$

for $1\le j\le n_1-1$,

$$b_{\sum_{k=1}^{i-1}n_k+j,\sum_{k=1}^{i-1}(2n_k-1)+2j-1}(t) = -\sqrt{p^i_j\lambda^i_j\bar{x}^i_j},$$
$$b_{\sum_{k=1}^{i-1}n_k+j,\sum_{k=1}^{i-1}(2n_k-1)+2j}(t) = -\sqrt{\left(1-p^i_j\right)\lambda^i_j\bar{x}^i_j},$$
$$b_{\sum_{k=1}^{i}n_k+1,\sum_{k=1}^{i-1}(2n_k-1)+2j}(t) = \sqrt{\left(1-p^i_j\right)\lambda^i_j\bar{x}^i_j},$$
$$b_{\sum_{k=1}^{i-1}n_k+j+1,\sum_{k=1}^{i-1}(2n_k-1)+2j-1}(t) = \sqrt{p^i_j\lambda^i_j\bar{x}^i_j},$$
$$b_{\sum_{k=1}^{i}n_k,\sum_{k=1}^{i}(2n_k-1)}(t) = -\sqrt{\lambda^i_{n_i}\bar{x}^i_{n_i}},$$
$$b_{\sum_{k=1}^{i}n_k+1,\sum_{k=1}^{i}(2n_k-1)}(t) = \sqrt{\lambda^i_{n_i}\bar{x}^i_{n_i}},$$

for $2\le i\le M-1$ and $1\le j\le n_i-1$. With the matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$, Proposition 4 derives the diffusion limit.

Proposition 4: On $t\in[0,T]$ for $T>0$,

$$\lim_{\eta\to\infty}\sqrt{\eta}\left\{\frac{X^{i,\eta}_j(t)}{\eta}-\bar{x}^i_j(t)\right\} = D^i_j(t) \text{ in distribution},$$

for $i=1,\cdots,M$, $j=1,\cdots,n_i$, where

$$\mathbf{D}(t) = \left(D^1_1(t),\ldots,D^1_{n_1}(t),\ldots,D^{M-1}_1(t),\right.$$
$$\left.\ldots,D^{M-1}_{n_{M-1}}(t),D^M_1(t)\right)'$$

is the solution to the following SDE.

$$d\mathbf{D}(t) = \mathbf{A}(t)\mathbf{D}(t)dt + \mathbf{B}(t)d\mathbf{W}(t), \mathbf{D}(0)=\mathbf{0}, \quad (8)$$

and $\mathbf{W}(t)$ is a $[(\sum_{i=1}^{M-1}2n_i-1)+1]$-dimensional standard Brownian motion. The covariance matrix of $\mathbf{D}(t)$, $\mathbf{\Sigma}(t)$, is the solution to the following ODEs.

$$\frac{d}{dt}\mathbf{\Sigma}(t) = \mathbf{A}(t)\mathbf{\Sigma}(t) + \mathbf{\Sigma}(t)\mathbf{A}(t)' + \mathbf{B}(t)\mathbf{B}(t)'. \quad (9)$$

Now, Lemma 2 derives the asymptotic distribution of the diffusion limit process, $\mathbf{D}(t)$.

Lemma 2: The diffusion limit process, $\mathbf{D}(t)$, is a Gaussian process; that is, for $t \in [0,T]$, $\mathbf{D}(t) = (D_1^1(t),\ldots,D_{n_1}^1(t),\ldots,D_1^{M-1}(t),\ldots,D_{n_{M-1}}^{M-1}(t),D_1^M(t))$ has a multivariate normal distribution with the mean being a $(\sum_{i=1}^{M-1} n_i + 1) \times 1$ zero vector, and the covariance matrix $\mathbf{\Sigma}(t)$.

Using the results in Propositions 3 and 4, and Lemma 2, for a large η, we can approximate $X_j^{i,\eta}(t)$ as follows.

$$X_j^{i,\eta}(t) \approx \eta \bar{x}_j^i(t) + \sqrt{\eta} D_j^i(t) \text{ in distribution},$$

and

$$\begin{aligned} \mathrm{E}\left[X_j^{i,\eta}(t)\right] &\approx \eta \bar{x}_j^i(t), \\ \mathrm{Var}\left[X_j^{i,\eta}(t)\right] &\approx \eta \mathrm{Var}\left[D_j^i(t)\right]. \end{aligned}$$

That is, for N units in the system, we obtain

$$\begin{aligned} \mathrm{E}\left[X_j^i(t)\right] &\approx N \bar{x}_j^i(t), \\ \mathrm{Var}\left[X_j^i(t)\right] &\approx N \mathrm{Var}\left[D_j^i(t)\right]. \end{aligned}$$

Note that the phases are virtual, and we are interested in the number of units at each degradation state. The number of units at State i is

$$X^i(t) = \sum_{j=1}^{n_i} X_j^i(t).$$

Then, we can easily calculate the mean vector and covariance matrix of $\mathbf{S}(t) = (X^1(t),\cdots,X^M(t))'$ using basic formulae on the mean and variance of the sum of random variables. The distribution of the system status at time t, $\mathbf{S}(t)$, approximately follows the multivariate normal distribution with a mean vector, $N[\sum_{j=1}^{n_1} \bar{x}_j^1(t),\cdots,\sum_{j=1}^{n_M} \bar{x}_j^M(t)]'$. The ith diagonal element, and the (i,i')th off-diagonal element of the covariance matrix of $\mathbf{S}(t)$ are

$$\begin{aligned} \mathrm{Var}\left[X^i(t)\right] &= \sum_{j=1}^{n_i} N \mathrm{Var}\left[D_j^i(t)\right] \\ &\quad + \sum_{j=1}^{n_i} \sum_{j'=1, j' \neq j}^{n_i} N \mathrm{Cov}\left[D_j^i(t), D_{j'}^i(t)\right], \\ \mathrm{Cov}\left[X^i(t), X^{i'}(t)\right] &= \sum_{j=1}^{n_i} \sum_{j'=1}^{n_{i'}} N \mathrm{Cov}\left[D_j^i(t), D_{j'}^{i'}(t)\right], \end{aligned}$$

respectively, for $i, i' = 1,\cdots,M$, $i \neq i'$, where $\mathrm{Var}[D_j^i(t)]$, $\mathrm{Cov}[D_j^i(t), D_{j'}^i(t)]$, and $\mathrm{Cov}[D_j^i(t), D_{j'}^{i'}(t)]$ are the elements of $\mathbf{\Sigma}(t)$ obtained from (9).

D. Implications

We summarize the advantages of our proposed approach for evaluating the system status in the non-Markovian processes. First, the proposed approach is general, and applicable to many applications. The phase-type distributions can approximate general distributions widely used in reliability studies, and can directly fit empirical transition data without explicitly finding transition distributions from the data [30]. Then, with the phase-type distribution approximation, we can apply fluid and diffusion limits to the non-Markovian degradation processes.

Next, by combining fluid and diffusion limits with the phase-type approximations, the computational complexity increases relatively slowly by the number of phases we use. In previous studies using phase-type distribution approximations, the usage of many phases (for better approximations) has been restricted due to the poor scalability (known as the state-space explosion problem). In our approach, the number of ODEs is $O(n^2)$ where n is the number of phases. Because the ODEs are only first order linear, we can obtain the solution in a few seconds even when we have more than 5 states, and 10 phases per each state. Therefore, the proposed approach is computationally efficient in evaluating the system status over its service life. To our best knowledge, the only way to evaluate the system status so far is to rely on simulations. Simulating the operations of a large-scale system over its service life can present computational issues, even with today's computing power. As such, our analytical approach can save computational time. The last but most important benefit of the presented method is that the results of the multivariate Gaussian distribution allow us to make a probabilistic assessment of the system status during the system's lifetime.

IV. Numerical Results

Motivated by recent construction of large-scale wind farms, we consider a wind farm with fifty turbines. We choose to use fifty units to evaluate the reliability assessment capability of the presented methodology in a mid-scale problem. As the system size increases, without additional computational burden, the proposed methods produce more accurate results due to its asymptotic properties.

In our implementations, we investigate the degradation process of gears inside a wind turbine gearbox. Because no actual data on multi-state degradation processes are available to us, based on the estimated lifetime distribution available in [35], we choose appropriate transition rates, and transition distributions for the Markovian process, and the non-Markovian process in Sections IV-A, and IV-B, respectively.

Fig. 5 illustrates the overall flow of our numerical study. The study starts with problem settings such as the number of states (we use $M = 4$), the number of turbines (we use $N = 50$), and transition time distributions (we use exponential and Weibull distributions for Markovian and non-Markovian processes, respectively). In the analytical approach, we first find the Coxian distributions for approximating the actual transition time distributions using the EM algorithm (for the Markovian case, this step is unnecessary). We then derive a system of ODEs as described in Sections II and III. We solve the system of ODEs with a numerical solver (in this study, we use GNU Octave, an open source numerical solver compatible with Matlab). To evaluate the accuracy of the proposed methods, we compare the results from the proposed analytical approach with the simulation results. In the simulation study, the computer code is written in C++ under the Windows environment. After creating states and turbine objects, we generate random numbers from the transition time distributions, and simulate the state transitions until

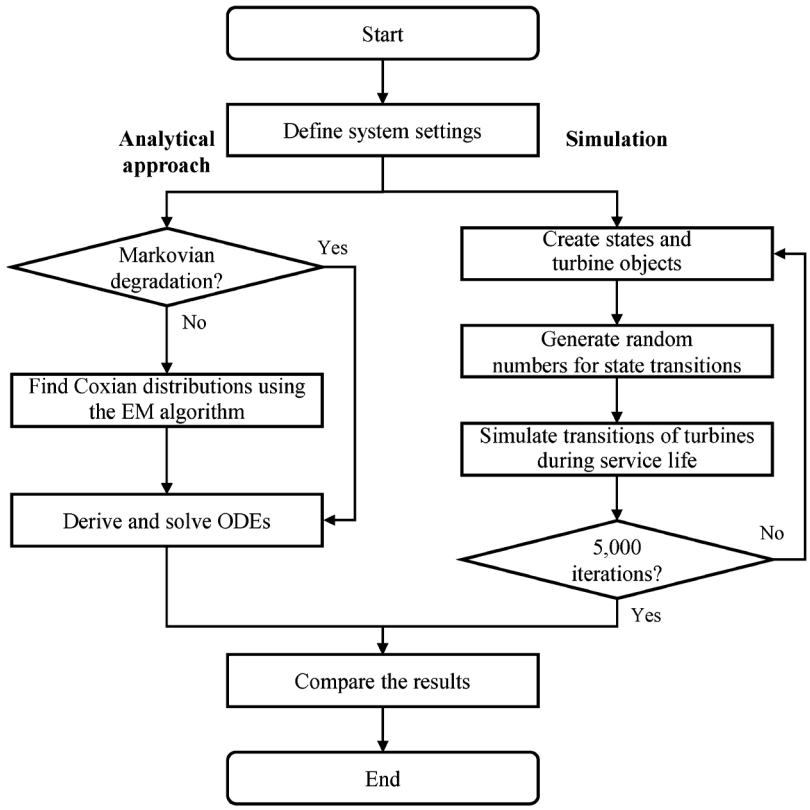

Fig. 5. Flow chart of the numerical study.

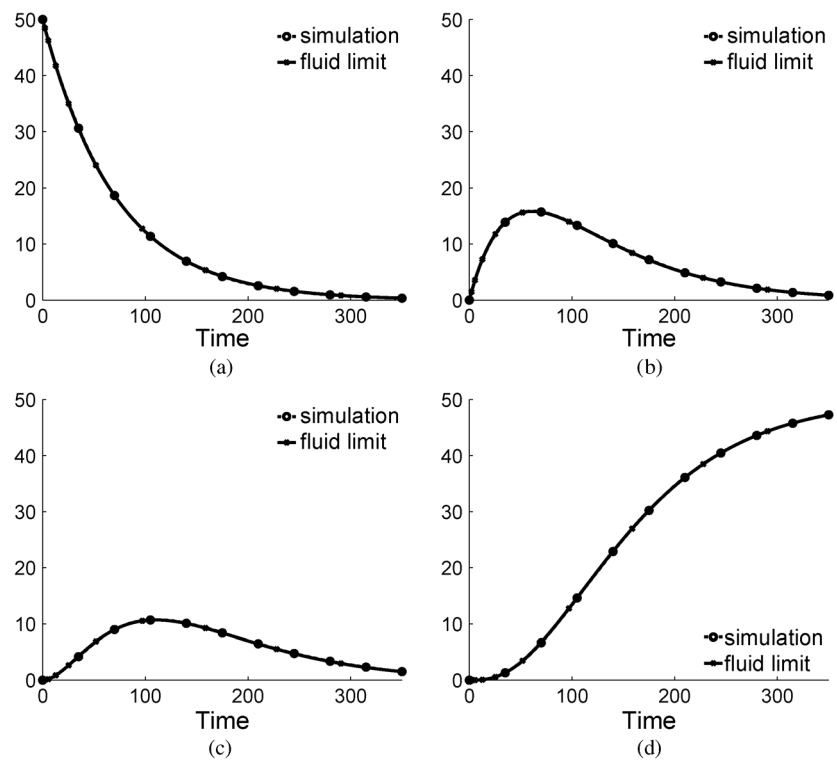

Fig. 6. Comparison of means from fluid limits with simulation results in the Markovian degradation process. (a) Average number of units at State 1. (b) Average number of units at State 2. (c) Average number of units at State 3. (d) Average number of units at State 4.

the predefined service life, T, is reached. We run 5,000 s-independent instances, and record the number of turbines at each state over time. Once the simulation is done, we compare the mean and covariance of the number of turbine gears at each state between the solution of the ODEs and the result from the simulation.

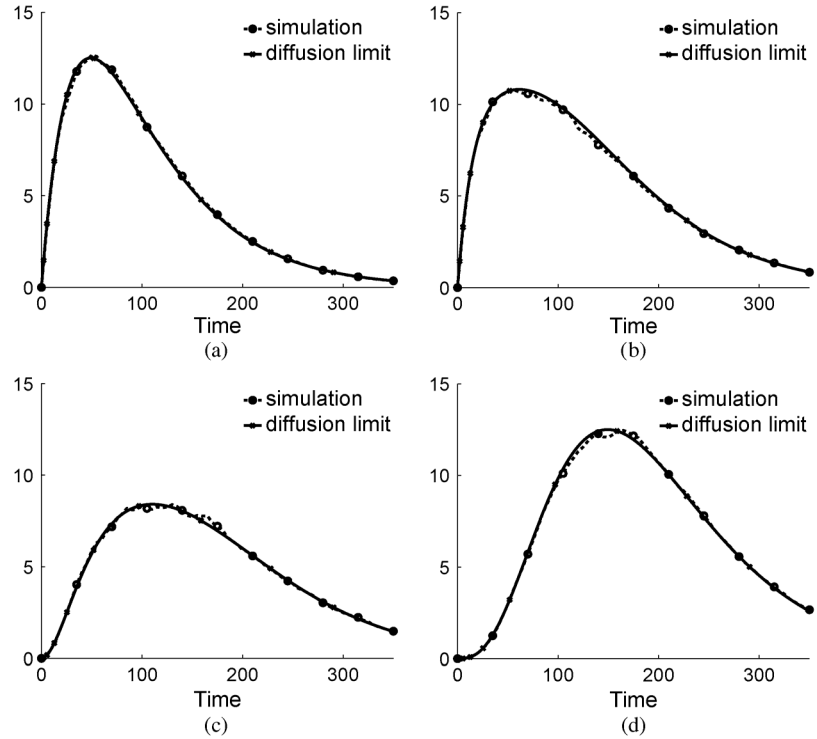

Fig. 7. Comparison of variances from diffusion limits with simulation results in the Markovian degradation process. (a) Variance of number of units at State 1. (b) Variance of number of units at State 2. (c) Variance of number of units at State 3. (d) Variance of number of units at State 4.

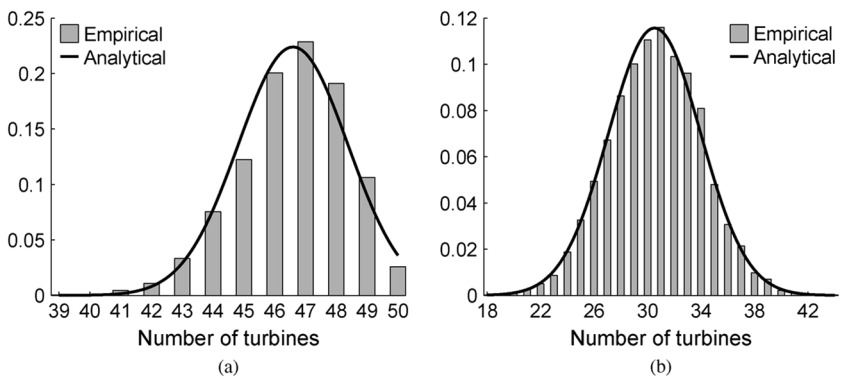

Fig. 8. Density of the number of units at State 1 in the Markovian degradation process: analytical Gaussian density from the proposed methods, and empirical density from simulations. (a) Density at $t = 5$. (b) Density at $t = 35$.

A. Markovian Degradation Process

We first evaluate the system status when wind turbine gears' degradation follows the Markovian process. We consider four degradation states (i.e., normal, alert, alarm, and failure) similar to the states used in [17], [18], [36]. According to Andrawus *et al.* [35], the average lifetime of wind turbine gears is 5,070 days (or about 169 months). In this implementation, we choose the Markovian transition rates as $\lambda_t^1 = 0.01409$, $\lambda_t^2 = 0.01878$, and $\lambda_t^3 = 0.02254$ per month, so that the resulting average lifetime with the four degradation states is consistent with that in [35].

Figs. 6 and 7 show that the means and variances from fluid and diffusion limits coincide with those from simulations. From Fig. 6, we can see that the expected number of units at State 1 (i.e., normal state) decreases over time. The expected number of gears at States 2 (i.e., alert state) and 3 (i.e., alarm state) peaks at around 70, and 110 months, respectively, whereas the number of gears at State 4 (i.e., failure state) increases over time. Fig. 7 indicates that the uncertainty is large when the age of gears is between 50 and 150 months.

Because the limit process is a Gaussian process, we can completely identify the distribution of the number of gears at each state with the obtained mean and variance. Fig. 8 shows the density of the number of gears at State 1 at two different times. In the very early age of operations, we observe a truncation in the right side of the density (see Fig. 8(a)), and there is a slight discrepancy between the analytical density obtained from the presented method and the empirical density from simulation runs. This discrepancy occurs because the analytical density is obtained from the limit process with the assumption of the infinite number of units; but, in our implementation, the number of gears is bounded between 0 and 50. We, however, can confirm the normality from Fig. 8(b) as the gears pass the early stage of operations.

TABLE I
GAUSSIAN DISTRIBUTION FOR THE NUMBER OF UNITS AT EACH STATE IN THE MARKOVIAN DEGRADATION PROCESS

Age (in months)	State 1 (Normal)	State 1 (Alert)	State 3 (Alarm)	State 4 (Failure)
50	N(25,12) [18-32]	N(16,11) [9-22]	N(7,6) [2-11]	N(3,3) [0-7]
100	N(12,9) [6-18]	N(14,10) [8-20]	N(11,8) [5-16]	N(13,10) [7-20]
150	N(6,5) [2-11]	N(9,7) [4-14]	N(10,8) [4-15]	N(25,12) [18-32]
200	N(3,3) [0-6]	N(5,5) [1-10]	N(7,6) [2-12]	N(35,11) [28-41]
250	N(1,1) [0-4]	N(3,3) [0-6]	N(4,4) [1-8]	N(41,7) [36-46]
300	N(1,1) [0-2]	N(2,2) [0-4]	N(3,3) [0-6]	N(45,5) [41-49]

Note: The numbers are rounded to the nearest integers. The numbers in the brackets in each cell denote the 95% confidence interval restricted between 0 and 50.

Table I summarizes the Gaussian distribution and 95% confidence intervals for the number of units at four states at several selected times. We can see that at $t = 150$, on average, 6, 9, 10, and 25 units are at States 1 through 4, respectively. It indicates that, even though the average lifetime of gears is about 169 months, preventive maintenance is necessary before 150 months to avoid a mass number of failures. Moreover, we can obtain the covariance matrix of the system status at time t, $N\Sigma(t)$, in the multivariate Gaussian distribution, which allows us to perform multivariate analysis. For instance, we can obtain the probability that more than 25 units (i.e., half of the entire units) are at the alarm or failure states at any selected time during a system's life.

B. Non-Markovian Degradation Process

Andrawus *et al.* [35] estimate the lifetime of gears using the Weibull distribution, $Weibull(2.5, 190)$. In our implementation in the non-Markovian degradation process, we assume that each transition follows the Weibull distribution, and consider four degradation states. In the three transition distributions, we use the same shape parameter, $\alpha_i = 2.5$, $i = 1, 2, 3$, as that in [35]. To find the scale parameters, we split the scale parameter (i.e., $\beta = 190$) in [35] into three portions, β_1, β_2, and β_3. To keep consistent with the parameters used in the Markovian process in Section IV-A, we choose the three scale parameters so that the expected state transition times are equal to those in Section IV-A. The time unit of the scale parameter in [35] was originally in days, but we convert it to months to relieve the imbalance between scale and shape parameters. The following Weibull distributions are used as the transition distributions.

- State 1 to 2: $Weibull(2.5, 80)$
- State 2 to 3: $Weibull(2.5, 60)$
- State 3 to 4: $Weibull(2.5, 50)$

Next, we approximate the chosen Weibull distributions with Coxian distributions. Table II shows the estimated parameters of the corresponding Coxian distributions with ten phases using the EM algorithm in [30]. Fig. 9, which compares the transition distribution from State 1 to 2 (i.e, $Weibull(2.5, 80)$) and the corresponding Coxian distribution, shows that the 10-phase Coxian distribution closely approximates the Weibull distribution.

TABLE II
ESTIMATED COXIAN DISTRIBUTION PARAMETERS

Phase	State transition					
	1 to 2		2 to 3		3 to 4	
	λ	p	λ	p	λ	p
1	0.1231	1.0000	0.1638	1.0000	0.1961	1.0000
2	0.1231	0.9994	0.1638	0.9994	0.1961	0.9993
3	0.1231	0.9642	0.1638	0.9646	0.1962	0.9649
4	0.1220	1.0000	0.1618	1.0000	0.1933	1.0000
5	0.1220	0.9563	0.1618	0.9501	0.1933	0.9443
6	0.1226	0.7324	0.1632	0.7386	0.1957	0.7442
7	0.1153	0.9947	0.1543	0.9923	0.1856	0.9907
8	0.1152	0.9995	0.1540	0.9986	0.1853	0.9975
9	0.1152	0.9972	0.1541	0.9926	0.1853	0.9871
10	0.1153	1.0000	0.1544	1.0000	0.1861	1.0000

TABLE III
GAUSSIAN DISTRIBUTION FOR THE NUMBER OF UNITS AT EACH STATE IN THE NON-MARKOVIAN DEGRADATION PROCESS

Age (in months)	State 1 (Normal)	State 2 (Alert)	State 3 (Alarm)	State 4 (Failure)
50	N(37,10) [31-43]	N(12,9) [6-18]	N(1,1) [0-2]	N(0,0) [0-0]
100	N(8,7) [3-14]	N(28,12) [21-35]	N(12,9) [6-17]	N(2,2) [0-5]
150	N(1,1) [0-2]	N(11,9) [6-17]	N(21,12) [14-27]	N(17,11) [11-24]
200	N(0,0) [0-0]	N(2,1) [0-4]	N(10,8) [4-15]	N(39,9) [33-45]
250	N(0,0) [0-0]	N(0,0) [0-1]	N(2,2) [0-4]	N(48,2) [46-50]

Note: The numbers are rounded to the nearest integers. The numbers in the brackets in each cell denote the 95% confidence interval restricted between 0 and 50.

We apply the limit processes to the resulting Coxian distributions, and compare the mean and variance of the number of gears at each state from the results of 5,000 s-independent simulation runs. Figs. 10 and 11 suggest that the results from the proposed analytical methods and simulation runs coincide, confirming that the proposed methods generate accurate evaluation for the system status in the non-Markovian degradation process.

Similar to the results in Section IV-A, in the densities of the number of turbine gears, we observe truncations in some cases due to the bound of the number of gears (see Fig. 12(a)). But, in most cases, Gaussian distributions obtained from the limit processes are complete without truncations, and the empirical densities from the simulation coincide with the Gaussian distributions (see Fig. 12(b)). Table III summarizes the distribution and 95% confidence intervals for the number of units at each state. We can see that the number of units at the alarm state increases up to $t = 150$ months, but the uncertainty increases as well. After 150 months, the number of units at the alarm state decreases, and many units transit to failure states (also see Figs. 10 and 11). Around 250 months, most units are in the failure state.

We also note that the distributions in the non-Markovian process differ from those in the Markovian process, even though the same average transition times are used in both processes. This result suggests that careful investigation is necessary to define the degradation transition distributions.

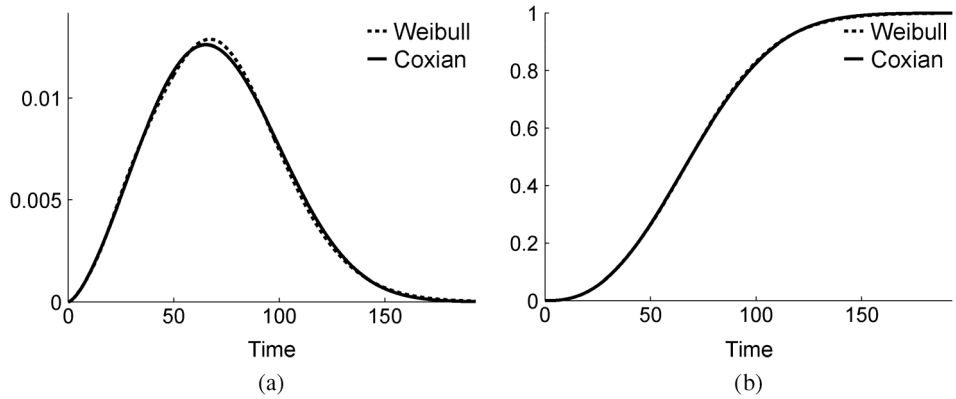

Fig. 9. Weibull (2.5, 80) and corresponding Coxian distributions. (a) Probability density function fitting. (b) Cumulative distribution function fitting.

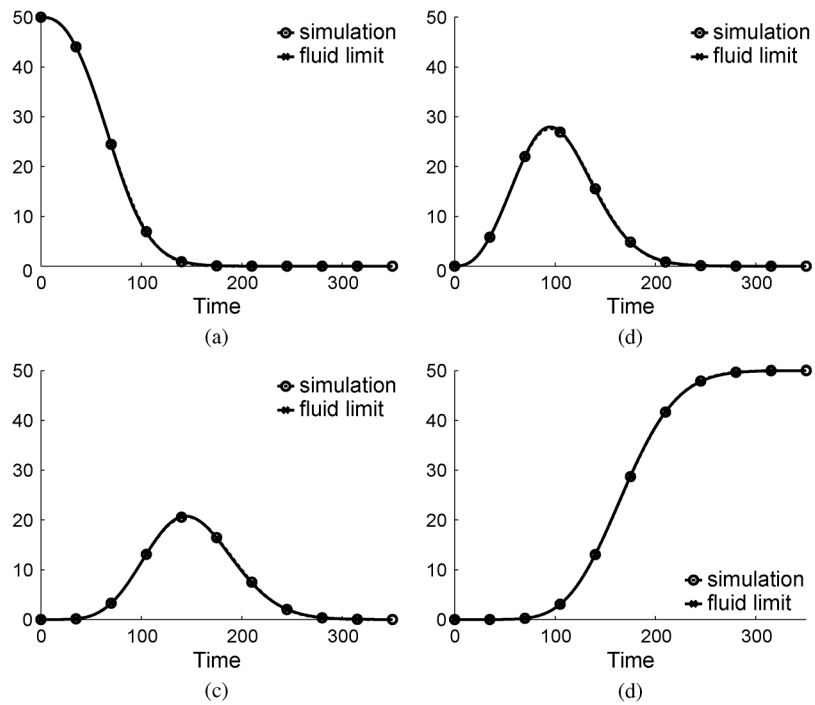

Fig. 10. Comparison of means from fluid limits with simulation results in the non-Markovian degradation process. (a) Average number of units at State 1. (b) Average number of units at State 2. (c) Average number of units at State 3. (d) Average number of units at State 4.

V. Conclusion

We present a general methodology to analyze the system reliability with a large number of identical units. In the literature, many degradation-based reliability studies focus on estimating a lifetime distribution for a system. From this conventional viewpoint, we switch to estimating the distribution of the number of units at multiple degradation states, and characterize the evolution of the system status.

We use fluid and diffusion limits, and derive the Gaussian distribution for the number of units at each state. When the units follow a non-Markovian degradation process, we integrate the phase-type approximation with fluid and diffusion limits. We believe this approach is a breakthrough to overcome computational difficulties in analyzing non-Markovian processes. Our implementation with fifty units suggests that the presented approach is accurate in characterizing the system dynamics. Due to its asymptotic properties, our methodology will generate more accurate results for larger-scale problems. For example, with a larger number of units, we will have fewer truncations in the Gaussian density that are caused by the bounded number of units. The presented approach is also computationally efficient to assess the reliability of large-scale systems.

The outcomes of this study will be useful for evaluating the system reliability at the design stage when historical operational data, or degradation testing data, are available for estimating the state transition distributions. One avenue of research we intend to pursue in the future is to update the system status, using real-time sensor data taken from individual units during system operations.

The proposed approach is most suitable when a system consists of many identical units that follow the same stochastic degradation processes. The representative applications include a wind farm with a large number of turbines where each turbine uses the same type of components (e.g., same gears for every turbine), and a solar park with identical solar panels. The

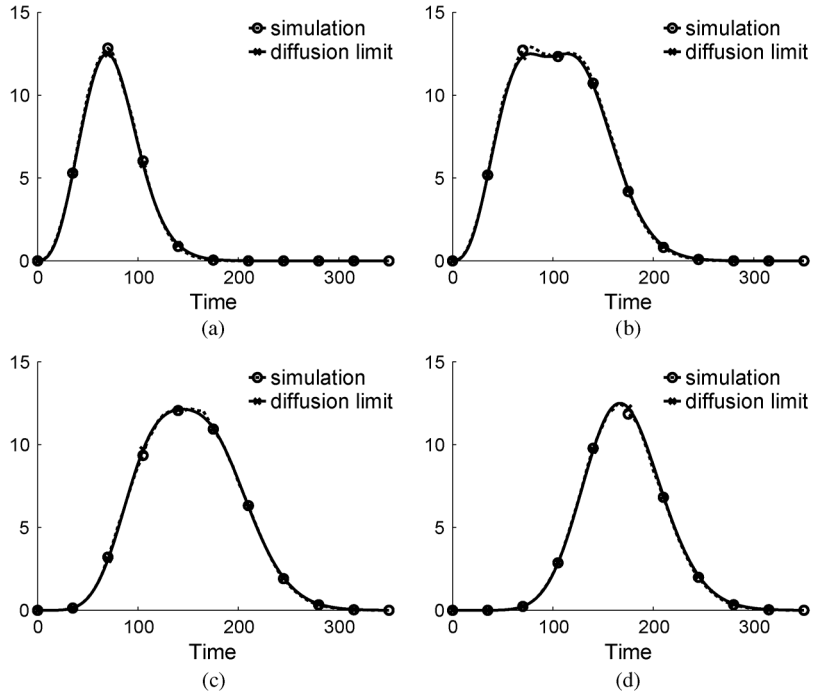

Fig. 11. Comparison of variances from diffusion limits with simulation results in the non-Markovian degradation process. (a) Variance of number of units at State 1. (b) Variance of number of units at State 2. (c) Variance of number of units at State 3. (d) Variance of number of units at State 4.

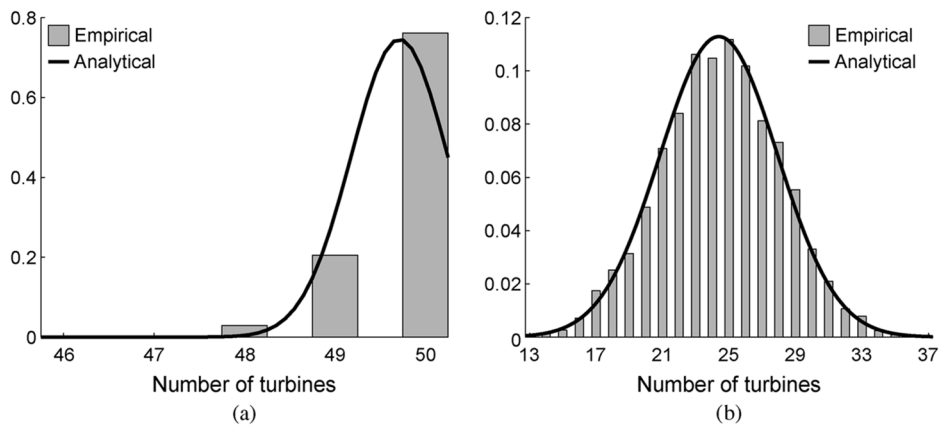

Fig. 12. Density of the number of units at State 1 in the non-Markovian degradation process: analytical Gaussian density from the proposed methods, and empirical density from simulations. (a) Density at time $t = 10$. (b) Density at time $t = 70$.

results of this study will be used to optimize management decision-making for those systems. For example, wind turbine manufacturers provide a short-term warranty (e.g., 12 months) after installations. After the warranty period, wind farm operators need to make post-warranty decisions, including the number of maintenance crew members, the warehouse size, and when and how many spare parts need to be ordered [37], [38]. These types of decisions are critical due to many logistic complexities such as the huge component size, and the heavy equipment needed to access turbines. We believe the results of this study will provide useful information for cost-effective operations and maintenance decision-making. We acknowledge that some other power generating systems use non-identical components even though their functions are similar. Our proposed approach has limitations for the systems with non-identical components, but we note that our future research will consider heterogeneous degradation processes for obtaining system-level reliability information. We also plan to extend the presented approach to a general case where the units' operational ages are heterogeneous due to different installation times, repair, replacement, etc.

APPENDIX

Proof of Proposition 1: A function $f(\cdot)$ is Lipschitz if $f(\cdot)$ satisfies the condition

$$|f(x) - f(y)| \leq K|x - y| \text{ for some constant } K.$$

Mandelbaum *et al.* [19] show that, if the rate functions are Lipschitz, and the initial values converge, the fluid limit holds in general cases (see Theorem 2.2 in [19]). In our case, the rate functions, which are the integrands in the integrals inside $Y_i(\cdot)$, are linear (Lipschitz), and satisfy the conditions in Theorem 2.2 in Mandelbaum *et al.* [19]. Therefore, the proposition is proved. □

Proof of Proposition 2: The rate functions, i.e., the integrands in the integrals inside $Y_i(\cdot)$, are linear and Lipschitz, which implies that they satisfy the conditions for the existence of diffu-

sion limits, following the results from Mandelbaum *et al.* [19] (see Theorem 2.3 in [19]). Therefore, the diffusion limits exist in the form of the solution to (6). Also, Arnold [39] shows that, for a linear SDE, the covariance matrix is the solution to (7) (see Theorem 8.2.6 in [39]). Because (6) is a linear SDE, we get the covariance matrix from (7). □

Proof of Lemma 1: According to the results from Arnold [39] (see Theorem 8.2.10 in Arnold [39]), $\mathbf{D}(t)$ is a Gaussian process iff $\mathbf{D}(0)$ is Gaussian or constant. In our case, $\mathbf{D}(0) = \mathbf{0}$, which is constant. Therefore, $\mathbf{D}(t)$ is a Gaussian process. □

Proof of Proposition 3: Because we use a finite number of phases, and the rate functions are Lipschitz, the proposition holds by applying the results from Mandelbaum *et al.* [19] (see Theorem 2.2 in [19]). □

Proof of Proposition 4: Our model uses a finite number of phases, and the rate functions are Lipschitz. Therefore, similar to the proof of Proposition 2, (8) holds by using the results from Mandelbaum *et al.* [19] (see Theorem 2.3 in [19]). Also, by applying the results from Arnold [39] (see Theorem 8.2.6 in [39]), the covariance matrix of $\mathbf{D}(t)$ can be obtained by solving the system of ODEs in (9). □

Proof of Lemma 1: Following the similar procedure as in Lemma 1, the proposition holds. □

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